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Uniform symbolic topologies and finite extensions ☆

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This paper is dedicated to the memory and many accomplishments of Hans-Bjørn Foxby

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ABSTRACT

We study the behavior of rings with uniform symbolic topologies with respect to finite extensions.

1. Introduction

The purpose of this note is to make some observations related to the following question.

Question 1.1. *Let $(R, \mathfrak{m})$ be a complete local domain. Does there exist a positive integer b such that $P^{(bn)} \subseteq P^n$, for all prime ideals $P \subseteq R$ and all $n \geq 1$?*

Here we write $P^{(t)}$ to denote the tth symbolic power of the prime ideal P. For any Noetherian domain R, when b as above exists, we shall say that R satisfies the *uniform symbolic topology property* on prime ideals. Uniform results of this type for regular rings were first given by Ein, Lazarsfeld and Smith in [5] and by Hochster and Huneke in [7]. These results prove that if R is a regular local ring containing a field, and d is the Krull dimension of R, then $P^{((d-1)n)} \subseteq P^n$, for all prime ideals $P \subseteq R$ and all $n \geq 1$. In [12], similar uniform results were proved for large classes of isolated singularities. In general, little is known: see the introduction to [12] for further discussion about this problem. Because a complete local domain is a finite extension of a regular local ring, we were led to consider how the uniform symbolic topology property behaves with respect to finite ring extensions. Thus, Question 1.1 would have a positive answer for complete local domains containing a field if, whenever $S \subseteq R$ is a finite extension of Noetherian domains, R has the uniform

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symbolic topology property on prime ideals if S has the uniform symbolic topology property on prime ideals. Unfortunately, we are not able to show that the uniform symbolic topology property for prime ideals lifts to a finite extension, but in Section 4, we are able to show that the property lifts for a certain family of ideals in the larger extension. In Section 3, we study the easier problem of the descent of the uniform symbolic topology property in a finite extension. In Section 2, we briefly review some important results concerning the equivalence of the adic and symbolic topologies of primes ideals, and more general ideals.

Remark 1.2. With no real extra effort, one could study potentially more precise uniform estimates by introducing two integers a, b in our definition of uniformity so that one can talk about the supremum of values $\frac{b}{a}$ for which $P^{(bn)} \subseteq P^{an}$ for all primes P and all n, whenever uniform bounds exist. It should be noted that the results in [5] and [7] cannot be improved in this asymptotic sense, as shown by Bocci and Harbourne in [2]. However, since we do not obtain very precise estimates, we have elected not to present the results in this manner, but to leave it as an interesting direction to pursue.

Remark 1.3. Many of the results in this paper concern uniform topologies for primes. One might wonder why we have restricted to prime ideals as opposed to, for example, self-radical ideals. (In fact, some of our results, e.g., Theorem 4.3, do apply to self-radical ideals.) However, the problem is the following: suppose one knows a given ring R has the uniform symbolic topology property with a uniform number b, and wishes to prove that R also has uniform symbolic topologies for self-radical ideals. Let $I = Q_1 \cap \cdots \cap Q_k$ be an intersection of primes. Then $I^{(bn)} = Q_1^{(bn)} \cap \cdots \cap Q_k^{(bn)} \subset Q_1^n \cap \cdots \cap Q_k^n$. But one still must compare, in a uniform way, the intersection of powers of the Q_i with powers of their intersection. It is obvious that $(Q_1^n \cap \cdots \cap Q_k^n)^k \subset (Q_1 \cap \cdots \cap Q_k)^n$ but k is not uniform for the ring. This limits the effectiveness of some of our methods.

We use several techniques in this paper. The ideas in the paper [6] play an important role in this paper. Standard facts on Galois extensions are used. The results and ideas behind the uniform Artin–Rees theorem are crucial. We combine theorems [9, Theorem 4.13] and [10, Theorem 5.4] into the following result:

Theorem 1.4. *Let R be a reduced ring satisfying one of the following conditions:*

(1) *R is essentially of finite type over either an excellent Noetherian local ring or $\mathbb{Z}$.*
(2) *R is a ring of characteristic p, and R is module finite over R^p.*
(3) *R is an excellent Noetherian ring which is the homomorphic image of a regular ring of finite Krull dimension such that for all primes P of R, R/P has a resolution of singularities obtained by blowing up an ideal.*

Then exists a positive integer ℓ such that for all ideals I of R and all $n > \ell$, $\overline{I^n} \subset I^{n-\ell}$.

Here we are using the notation $\overline{J}$ to denote the integral closure of an ideal J.

The previous results lead to the following definition.

Definition 1.5. We say that a ring R is *acceptable* if R and all finite extension rings of R satisfy one of the following conditions: they are either essentially of finite type over an excellent Noetherian local ring or the integers, have characteristic p and are F-finite, or are an excellent Noetherian ring with infinite residue fields such that all domains which homomorphic images have a resolution of singularities obtained by blowing up an ideal. Observe that if R is as in one of the first two cases, then R is acceptable since rings of the first two types are automatically preserved under finite extensions.

For unexplained terminology, we refer the reader to the book [4].

2. Equivalence of topologies

In this section we recall some relevant results concerning equivalence of various ideal topologies. We then use these results to show that Question 1.1 is well-posed.

Let P be a prime ideal. We choose an element $s \notin P$, such that s is in every embedded prime of every P^n ($n \geq 2$). Such an s exists by [3] which proves that the number of associated primes of powers of P is finite. Then for $J := sR$, it is not difficult to see that $P^{(n)} = P^n : \langle J \rangle$, for all $n \geq 1$. Recall that for ideals $I, J \subseteq R$,

$$I : \langle J \rangle := \bigcup_{n \geq 1} (I : J^n).$$

Thus, any statements about the equivalence of the topologies defined by the sets $\{I^n\}$ and $\{I^n : \langle J \rangle\}$ apply to the adic and symbolic topologies defined by a prime ideal. Numerous authors have studied the equivalence of these various topologies, e.g., see [14,15,8]. A definitive answer was given in the following theorem of Schenzel (see [14]):

Theorem 2.1 *(Schenzel). Let A be a Noetherian ring and $I, J \subseteq A$ two ideals. Then the following are equivalent.*

(1) *The $\{I^n : \langle J \rangle\}$ topology is equivalent to the I-adic topology.*
(2) $\dim(\widehat{R_Q}/(I\widehat{R_Q} + z)) > 0$*, for all prime ideals $Q \in A(I) \cap V(J)$, and prime ideals $z \in \operatorname{Ass}(\widehat{R_Q})$.*

Here, $A(I)$ is the (stable) set of associated primes of high powers of I. Note that in Schenzel's theorem, to say that the $\{I^n : \langle J \rangle\}$ topology is equivalent to the $\{I^n\}$ topology means that for all $n \geq 1$, there exists t (depending on n) such that $I^t : \langle J \rangle \subseteq I^n$. Thus, if $P \subseteq R$ is a prime ideal, it follows from the first paragraph of this section that the $\{P^{(n)}\}$ topology is equivalent to the $\{P^n\}$ topology if and only if for all $Q \in \bigcup_{n \geq 2} \operatorname{Ass}(R/P^n)$, $Q \neq P$, $\dim(\widehat{R_Q}/(P\widehat{R_Q} + z)) > 0$, for all $z \in \operatorname{Ass}(\widehat{R_Q})$. In particular, the topologies are equivalent if each R_Q is analytically irreducible.

In the mean time, in answer to a question of Schenzel, Irena Swanson was able to prove the following beautiful result concerning equivalent topologies [16].

Theorem 2.2 *(Swanson). Let R be a Noetherian ring and I, J ideals. Then the $\{I^n\}$ and $\{I^n : \langle J \rangle\}$ topologies are equivalent if and only if there exits $c \geq 1$, so that for all $n \geq 1$, $I^{cn} : \langle J \rangle \subseteq I^n$.*

It is important to observe that the integer c in the statement of Swanson's theorem depends *a priori* on I. In particular, the theorem applies to symbolic powers: If $P \subseteq R$ is a prime ideal, and the P-symbolic and P-adic topologies are equivalent, then there exists $c \geq 1$ so that $P^{(cn)} \subseteq P^n$, for all n. The results of Ein–Lazarsfeld–Smith and Hochster–Huneke referred to in the introduction show that for a regular local ring containing a field, a uniform c may be taken independent of the prime ideal P. Question 1.1 asks for a similar uniformity for complete local domains.

Remark 2.3. For ideals $I, J \subseteq R$, suppose we have $d, t \geq 1$ such that $I^{dn} : \langle J \rangle \subseteq I^{n-t}$, for all $n > t$. Then an elementary induction argument shows that for $c := d(t + 1)$, $I^{cn} : \langle J \rangle \subseteq I^n$, for all $n \geq 1$.

The following proposition shows that complete local domains have the property that the adic and symbolic topologies are linearly equivalent for all prime ideals, i.e., that Question 1.1 is well-posed.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a complete local domain and let $P \subseteq R$ be a prime ideal. Then the $\{P^{(n)}\}$ topology is linearly equivalent to the $\{P^n\}$ topology. In particular, there exists $c > 0$ (depending on P) such that $P^{(cn)} \subseteq P^n$, for all $n \geq 1$.*

Proof. It suffices to prove the second statement. Let S denote the integral closure of R and set $I := \operatorname{rad}(PS)$. Since S is an excellent normal domain, it is locally analytically normal, so the completion of S_Q is a domain for all primes Q. In particular, by Schenzel's theorem, the $\{I^{(n)}\}$ topology is equivalent to the $\{I^n\}$ topology. Here we are writing $I^{(n)}$ for $I^n_U \cap S$, where $U := S \backslash Q_1 \cup \cdots \cup Q_r$, for $Q_1, \ldots, Q_r$ the primes in S lying over P, so that $I = Q_1 \cap \cdots \cap Q_r$. Thus, by 2.2, $P^{(kn)} \subset I^{(kn)} \subset I^n$ for some fixed k and all $n \geq 1$. On the other hand, there is an e such that $I^e \subset PS$, so that

$$P^{(ken)} \subset P^n S \cap R \subset P^{n-l},$$

for some l by the Artin–Rees lemma. By Remark 2.3, taking $c := ke(l + 1)$ gives $P^{(cn)} \subseteq P^n$, for all n. □

3. Descent of the uniform symbolic topology property

The situation in this section is the easier one of those considered in this paper: if $R \subset S$ is a finite extension of Noetherian rings such that S has the uniform symbolic topology property on prime ideals, then when does R also have uniform symbolic topology property on primes? The more difficult situation of trying to prove the reverse will be treated in the next section. We begin with an easy but crucial observation.

Observation 3.1. Let $R \subset S$ be a finite extension of Noetherian domains such that R is integrally closed and the degree of the extension of fraction fields is e. If q is a prime in R, then

$$(\sqrt{qS}\,)^e \subset \overline{qS}.$$

To see this, let $x \in \sqrt{qS}$. By [1, Lemma 5.14], x satisfies a monic polynomial over R with sub-monic coefficients in q. So, by [1, Proposition 5.15] the sub-monic coefficients of the minimal polynomial of x over the fraction field of R, which has degree at most e, are in q. It follows from this that $x^e \in qS$. Now take $x_1, \ldots, x_e \in \sqrt{qS}$. By what we have just shown, $(x_1 \cdots x_e)^e \in q^e S$. It follows that $x_1 \cdots x_e \in \overline{qS}$, and therefore $(\sqrt{qS}\,)^e \subseteq \overline{qS}$.

In the preceding observation, one may in fact conclude that $(\sqrt{qS}\,)^e \subseteq qS$ if $e!$ is invertible. This is because in this case, the eth powers of all elements in an ideal generate the eth power of the ideal. See page 109, 1.11, in [11].

Theorem 3.2. *Let $R \subset S$ be a finite integral extension of Noetherian domains. Assume that R is acceptable and integrally closed. There exists an integer r, depending only on the extension $R \subset S$, such that if Q is a prime in S, $q = R \cap Q$, and $Q^{(bn)} \subset Q^n$, for some fixed b and for all $n \geq 1$, then $q^{(rbn)} \subset q^n$ for all $n \geq 1$.*

Proof. Suppose $Q \subseteq S$ is a prime ideal, $q := Q \cap R$, and $Q^{(bn)} \subseteq Q^n$, for all $n \geq 1$. To prove the theorem it is enough to show there exists a positive integer r, independent of q, such that for all positive integers n,

$$Q^{rn} \cap R \subset q^n.$$

For then $q^{(rbn)} \subset Q^{(rbn)} \cap R \subset Q^{rn} \cap R \subset q^n$, as needed.

We move this point to a separate lemma.

Lemma 3.3. *Let R be an integrally closed acceptable Noetherian domain with fraction field K, and let S be a finite extension of R with fraction field F. There exists an integer r such that for all primes q of R and all P lying over q,*

$$P^{rn} \cap R \subset q^n$$

for all $n \geq 1$.

Proof. Let $q \subseteq R$ be a prime ideal and $P \subseteq S$ a prime lying over q. We first note that we may assume that S is integrally closed. Indeed, if P_0 is a prime in the integral closure of S lying over P, then if $P_0^{rn} \cap R \subseteq q^n$, for all $n \geq 1$, it clearly holds that $P^{rn} \cap R \subseteq q^n$, for all n. Thus, we may replace S by its integral closure, or, without loss of generality, assume that S is integrally closed. Note that S is still finite over R since R is excellent.

Now, let $K \subseteq L \subseteq F$ be the intermediate field which satisfies $K \subseteq L$ is separable and $L \subseteq F$ is purely inseparable. Let T denote the integral closure of R in L, so $R \subseteq T \subseteq S$. For q and P as above, suppose $Q := P \cap T$. If $P^{cn} \cap T \subseteq Q^n$, for all $n \geq 1$ and $Q^{dn} \cap R \subseteq q^n$, for all $n \geq 1$, then for $r = cd$, it follows that $P^{rn} \cap R \subseteq q^n$, for all $n \geq 1$. Thus, we may begin again assuming that S is integrally closed and prove that the conclusion of the lemma holds if either F is separable over K or F is purely inseparable over K.

Suppose first that F is separable over K. Then, as in the first paragraph above, since we are free to enlarge S, we may further assume that F is Galois over K (by replacing S by its integral closure in a normal extension of F).

Let $q \subseteq R$ be a prime ideal and $P \subseteq S$ a prime lying over q. Suppose that

$$\operatorname{rad}(qS) = P_1 \cap \cdots \cap P_t.$$

By [17, page 43, Exercise 2.8], the number of prime ideals in S lying over q is at most the degree $e = [F : K]$. Thus the number t of prime ideals minimal over qS is uniformly bounded. Let $a \in P^{e^2 n} \cap R$. Note that P is among the P_i and that the P_i are permuted transitively from P under the action of the Galois group by [18]. Then $a \in P_i^{e^2 n}$ for all i. Therefore,

$$a^e \in \left(a^t\right) \subset P_1^{e^2 n} \cdots P_t^{e^2 n}.$$

By Observation 3.1, $\operatorname{rad}(qS)^e \subset \overline{qS}$. Therefore we obtain

$$a^e \in (P_1 \cdots P_t)^{e^2 n} \subset (P_1 \cap \cdots \cap P_t)^{e^2 n} = \operatorname{rad}(qS)^{e^2 n} \subset \overline{q^{en} S}.$$

Thus, $a^e \in \overline{q^{en} S} \cap R = \overline{q^{en}}$. Therefore, $a \in \overline{q^n}$. By Theorem 1.4, there exists $l > 0$ such that $a \in q^{n-l}$, for all $n > l$. Thus, for $n > l$, $P^{e^2 n} \cap R \subseteq q^{n-l}$. By the reasoning employed in Remark 2.3, if we take $r := e^2(l+1)$, we obtain $P^{rn} \cap R \subseteq q^n$, for all $n \geq 1$.

Now suppose that F is purely inseparable over K. Then P is the only prime ideal of S lying over q, so $P = \operatorname{rad}(qS)$. Thus, if we start with $a \in P^{en} \cap R$, then, by Observation 3.1,

$$a^e \in \left(P^{en}\right)^e \cap R \subseteq \overline{q^{en} S} \cap R = \overline{q^{en}}.$$

Therefore, $a \in \overline{q^n}$. Thus, as before, if we take $r := e(l+1)$, we have $P^{rn} \cap R \subseteq q^n$, for all $n \geq 1$.

Since l and e are independent of q, this completes the proof of the lemma and hence also the proof of Theorem 3.2. $\square$

Corollary 3.4. *Let $R \subset S$ be a finite integral extension of Noetherian domains. Assume that R is acceptable and is integrally closed. If S has the uniform symbolic topology property for prime ideals then so does R.*

The following special case is worth recording:

Corollary 3.5. *Let S be a regular local ring containing a field of characteristic* 0*, let G be a finite group of automorphisms of S, and set R equal to the ring of invariants, S^G. If R is acceptable, then R has the uniform symbolic topology property on prime ideals.*

4. Ascent of the uniform symbolic topology property

In this section we prove two theorems regarding ascent of uniform topologies. The first does the purely inseparable case, which is easier, while the second theorem deals with separable extensions. Both theorems deal with the nilradicals of extensions of prime ideals. We combine these results in the case both rings are integrally closed in a final corollary.

Theorem 4.1. *Let $R \subset S$ be a finite integral extension of Noetherian domains. Assume that R is acceptable, is integrally closed with fraction field K, and that the field of fractions L of S is a purely inseparable extension of K. There exists an integer r, depending only on the extension $R \subset S$, such that if $I = \sqrt{I}$ is a self-radical ideal in R and $J = \sqrt{IS}$ and if $I^{(bn)} \subset I^n$, for some fixed b and for all n, then $J^{(rbn)} \subset J^n$ for all n.*

Proof. Let e be the degree of the field extension $K \subset L$. Let $x \in J^{(be(\ell+1)n)}$, where ℓ is chosen so that $\overline{\mathfrak{a}^n} \subset \mathfrak{a}^{n-\ell}$ for all ideals $\mathfrak{a}$ of S by using Theorem 1.4. Let Q be any prime minimal over J, and let $q = Q \cap R$. Since $x \in Q^{(be(\ell+1)n)}$, a result of Hochster in [6, Proposition 3.6] gives that $x^e \in q^{(be(\ell+1)n)}$. Since any minimal prime q over I is the contraction of a minimal prime over J, it follows that

$$x^e \in I^{(be(\ell+1)n)} \subset I^{(\ell+1)en} \subset J^{(\ell+1)en}.$$

Here we are using the fact that $I^{(be(\ell+1)n)}$ is the intersection of the $q^{(be(\ell+1)n)}$, with q ranging over the minimal primes of I. Thus, $x \in \overline{J^{(\ell+1)n}}$. Hence,

$$J^{(be(\ell+1)n)} \subset \overline{J^{(\ell+1)n}} \subset J^{(\ell+1)n-\ell} \subset J^n,$$

for all $n \geq 1$, and we may set $r = e(\ell+1)$. □

We have the following corollary which shows that the uniform symbolic topology property lifts in a purely inseparable extension.

Corollary 4.2. *Let $R \subset S$ be a finite integral extension of Noetherian domains. Assume that R is acceptable, is integrally closed with fraction field K, and that the field of fractions L of S is a purely inseparable extension of K. If R has the uniform symbolic topology property on primes ideals, then S has the uniform symbolic topology property on prime ideals.*

Proof. For any prime ideal $Q \subseteq S$, Q is the unique prime ideal of S lying over $q := Q \cap R$. Thus, $Q = \sqrt{qS}$. The result now follows from the previous theorem. □

Theorem 4.3. *Let $R \subset S$ be an integral extension of Noetherian domains. Assume that R is integrally closed with fraction field K and that the field of fractions L of S is a finite separable extension of K. Further assume that R is acceptable. There exists an integer $r \geq 1$, depending only on the extension of rings $R \subset S$ with the following property: given a prime q in R and an integer $b \geq 1$ such that for all $n \geq 1$,*

$$q^{(bn)} \subset q^n,$$

then for all $n \geq 1$,

$$J^{(rbn)} \subset J^n$$

where $J = \sqrt{qS}$.

Proof. We first reduce the theorem to the case in which L is Galois over K. Let F be the normal closure of L over K and let T be the integral closure of R in F. Assume we have proved the theorem for the extension $R \subset T$ with $\mathfrak{a} := \sqrt{qT}$ and $t > 0$ independent of q such that $\mathfrak{a}^{(tbn)} \subseteq \mathfrak{a}^n$, for all $n \geq 1$. Let $f = [F : K]$ and $a \in J^{(tfbn)}$. There exists an $s \in S$, not in any minimal prime over J, such that $sa \in J^{tfbn} \subset \mathfrak{a}^{tfbn}$. Since s is not in any minimal prime of J, it follows that s is also not in any minimal prime of $\mathfrak{a}$. Hence $a \in \mathfrak{a}^{(tfbn)} \subset \mathfrak{a}^{fn}$, by using our assumption. Observation 3.1 then gives that $a \in \overline{(qT)^n}$, and thus $a \in \overline{q^n S} = S \cap \overline{q^n T}$. Taking $l > 0$ as in Theorem 1.4 (applied to S), it follows $a \in q^{n-l}S$, for all $n > l$. Therefore, $J^{(tfbn)} \subseteq J^{n-l}$ for $n > l$. Setting $r := tf(l + 1)$, Remark 2.3 yields $J^{(rbn)} \subseteq J^n$, for all $n \geq 1$. Thus it suffices to prove the Galois case. In the remainder of the proof we assume that L is a Galois extension of K.

Let e be the degree of the field extension $K \subset L$, and let ℓ be such that for all ideals I of R and S, $\overline{I^n} \subset I^{n-\ell}$ for all $n > \ell$. We claim that the choice $r = e(e+1)(\ell+1)$ satisfies the conclusion of the theorem.

By [3] or [13], the number of associated primes of J^n is finite as n varies, and therefore so is the set of primes in R which are contractions of these primes. Similarly, the number of associated primes of q^n is finite as n varies. Let Λ be the union of these two finite set of primes after removing q. Choose $f \in R$ in the intersection of all primes in Λ and such that $f \notin q$. It follows that

$$J^{(n)} = J^n : \langle f \rangle$$

and

$$q^{(n)} = q^n : \langle f \rangle.$$

Set $r := e(e+1)(\ell+1)$, let $u \in J^{(rbn)}$ and write $F(x) = x^e + t_1 x^{e-1} + \cdots + t_e = (x - \sigma_1(u)) \cdots (x - \sigma_e(u))$, where $\sigma_1, \ldots, \sigma_e$ are the elements of the Galois group of L over K. Note that the t_j's are, up to a sign, symmetric polynomials in $\sigma_1(u), \ldots, \sigma_e(u)$, Choose a positive integer s such that $f^s \cdot u \in J^{rbn}$. Then for all $1 \leq i \leq e$,

$$f^s \cdot \sigma_i(u) = \sigma_i\big(f^s \cdot u\big) \in \sigma_i(J)^{rbn} = J^{rbn}$$

as J is the nilradical of qS by hypothesis and is therefore fixed by the Galois group. Thus, $f^{se} \cdot t_j \in J^{rbn}$, for all j. Since $J^e \subset \overline{qS}$ by Observation 3.1, we have for all $n \geq 1$,

$$f^{se} \cdot t_j \in J^{rbn} \cap R \subset \overline{q^{dbn} S} \cap R \ = \overline{q^{dbn}},$$

for $d = (e+1)(\ell+1)$. Hence, by choice of ℓ,

$$f^{se} \cdot t_j \in q^{dbn-\ell} \subseteq q^{e(\ell+1)bn},$$

for $n \geq 1$.

Thus, for $1 \leq j \leq e$,

$$t_j \in q^{e(\ell+1)bn} : \langle f \rangle = q^{(e(\ell+1)bn)} \subset q^{e(\ell+1)n}.$$

Since $u^e + t_1 u^{e-1} + \cdots + t_e = F(u) = 0$, we obtain

$$u^e \in q^{e(\ell+1)n} S.$$

Therefore $u \in \overline{q^{(\ell+1)n} S} \subseteq q^{(\ell+1)n-\ell} S \subseteq q^n S \subseteq J^n$, for all $n \geq 1$. $\square$

We summarize the ascent of the uniform symbolic topology property in the following corollary:

Corollary 4.4. *Let $R \subset S$ be a finite integral extension of Noetherian domains. Assume that R is acceptable and R is integrally closed in its field of fractions K and that the field of fractions L of S is a separable extension of K. If R has the uniform symbolic topology property on prime ideals, then S has the uniform symbolic topology property for all ideals which are the radical of extensions of prime ideals of R. In particular, the uniform symbolic topology property holds for the set of prime ideals $Q \subseteq S$ for which Q is the unique prime ideal lying over $Q \cap R$.*

Finally, we can combine both theorems of this section to give the general case, assuming that S is also integrally closed.

Corollary 4.5. *Let $R \subset S$ be a finite integral extension of Noetherian domains. Assume that R is acceptable and that R (respectively S) is integrally closed in its field of fractions, K (respectively L). If R has the uniform symbolic topology property on prime ideals, then S has the uniform symbolic topology property for all ideals which are the radical of extensions of prime ideals of R. In particular, the uniform symbolic topology property holds for the set of prime ideals $Q \subseteq S$ for which Q is the unique prime ideal lying over $Q \cap R$.*

Proof. We let F be the separable closure of K in L, so that F is separable over K, and L is purely inseparable over F. Let T be the integral closure of R in F. By Corollary 4.4, it follows that T has the uniform symbolic topology property for all ideals which are the radical of extensions of prime ideals of R. Since S is integrally closed, it contains T, and we may apply Theorem 4.1 to the pair $T \subset S$ to finish the proof. $\square$

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