Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 322 (2009) 3194–3211
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On the support of local cohomology

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Received 2 August 2007
Available online 2 April 2008
Communicated by Melvin Hochster

Dedicated to Professor Paul Roberts on the occasion of his sixtieth birthday

Abstract

We examine the question of whether the support of an arbitrary local cohomology module of a finitely generated module over a Noetherian ring with support in a given ideal must be closed in the Zariski topology. Several results giving an affirmative answer to this question are given; in particular, we show that the support is closed when the given ideal has cohomological dimension at most two or the ambient ring is local of dimension at most four.

Keywords: Local cohomology; Support; Associated primes

1. Introduction

Examples of A. Singh [22], M. Katzman [12], and later I. Swanson and Singh [23], show that the set of associated primes of a local cohomology module $H_I^i(R)$ of a Noetherian (commutative) ring R with support in an ideal I can be infinite. However, it remains an open question whether the sets of primes minimal in the support of such local cohomology modules are always finite.

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[1] The first author gratefully acknowledges the support of the National Science Foundation, grant DMS-0244405.
[2] The third author was supported by a grant from the University of Nebraska-Lincoln Research Council, and would like to thank the University of Kansas for its hospitality during his visit in the spring of 2006.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.02.021

This is equivalent to asking whether the support of local cohomology modules of Noetherian rings (or modules) must be Zariski-closed subsets of Spec R. Clearly, this question is of central importance in the study of cohomological dimension and understanding the local–global properties of local cohomology.

The answer to this question is known in several cases. Of course, if the set of associated primes of a given local cohomology module is finite then its support is closed. Thus, $H_I^i(R)$ has closed support if R is a regular local ring containing a field [10,15] or if R is an unramified regular local ring of mixed characteristic [16]. Additionally, the set of associated primes of $H_I^i(R)$ is finite if $i \in \{0, 1\}$, $i = \mathrm{depth}_I R$, $i = \dim R$, or (in the case R is local) $i = \dim R - 1$ [17]. The first non-finitely generated local cohomology has finitely many associated primes by [2] and independently [14].

Of particular interest is whether "top" local cohomology modules always have closed support. (We say $H_I^i(R)$ is a 'top' local cohomology module if $H_I^j(R) = 0$ for all $j > i$ and $H_I^i(R) \neq 0$.) Rotthaus and Sega [21] proved that the support of the top local cohomology of the irrelevant ideal is always closed in a standard graded Noetherian ring.[3] Katzman, in an argument attributed to G. Lyubeznik, proves that if R is a ring of positive prime characteristic and I is an ideal generated by n elements, then the support of $H_I^n(R)$ is closed [13]. Whether the corresponding result holds for rings containing a field of characteristic zero is an open question. If I is a three-generated grade two perfect ideal in an equicharacteristic zero local ring, we do not know whether the support of $H_I^3(R)$ is closed. In fact, as we will prove in Corollary 6.4 this is in some sense the main case for understanding when the support of $H_I^n(R)$ is closed (see also comments below).

All of these results provide positive answers in special cases to the following question:

Question 1.1. *Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal generated by n elements. Is the support of $H_I^n(M)$ is closed?*

Using a theorem of Gruson, we can show that it suffices to give a positive answer to the question in the case $M = R$ (Proposition 2.1). Using ideas of M. Hellus [7], we show that if the answer is positive for $n = 3$ then it is for $n \geqslant 4$ as well (Corollary 5.3). The question is trivial for $n = 1$. The case $n = 2$ and $M = R$ follows from a result of Hochster [8, Corollary 6.11], which we generalize in the following theorem (Theorem 2.4):

Theorem 1.2. *Let R be a Noetherian ring and I an ideal such that $H_I^i(R) = 0$ for $i > 2$. Then the support of $H_I^2(M)$ is closed for all finitely generated R-modules M.*

Unfortunately, our techniques cannot work for higher cohomological dimension without substantial modifications, due to an example of Paul Roberts, which we describe later in detail. See [20].

In Section 3 we prove that the support of $H_I^i(M)$ is closed for all i and all ideals I whenever M is a finitely generated module over a local ring of dimension at most four. In Section 4 we prove a generalization of Lyubeznik's characteristic p result mentioned above.

If R contains a field of characteristic zero, we prove the following compelling result (Theorem 6.1): Let I be an ideal generated by n elements, where $n \geqslant 6$. Then there exists a 2×3 matrix

[3] Earlier Katzman [13] proved that if R is an $\mathbb{N}$-graded Noetherian ring such that R_0 is the homomorphic image of a domain and R_+ is generated by n homogeneous forms, then $H_{R_+}^n(R)$ has closed support.

A with entries from R such that $H_I^n(R) \cong H^3_{I_2(A)}(R)$ where $I_2(A)$ is the ideal generated by the 2×2 minors of A. Moreover, if $\operatorname{depth}_I R \geqslant 2$ then one may assume $\operatorname{depth}_{I_2(A)} R \geqslant 2$ and hence $\operatorname{pd}_R R/I_2(A) = 2$. This theorem has some surprising consequences, particularly in connection to an example of Hartshorne (Example 6.6).

Throughout this paper all rings are assumed to be commutative with identity and all modules unital. For an R-module M, we let $\operatorname{Ass}_R M$ and $\operatorname{Supp}_R M$ denote the set of associated primes and support, respectively, of M. We let $\operatorname{Min}_R M$ denote the set of primes minimal in $\operatorname{Supp}_R M$ (or equivalently, in $\operatorname{Ass}_R M$). The reader should consult either [4] or [3] for any unexplained notation or terminology.

2. Cohomological dimension two

Let R be a Noetherian ring, I an ideal, and M a finitely generated R-module. As noted in the introduction, it is well known that $\operatorname{Ass}_R H_I^i(M)$ is finite for $i \in \{0, 1\}$, and thus $\operatorname{Supp}_R H_I^i(M)$ is closed in these cases. An example of Katzman [12] shows that $\operatorname{Ass}_R H_I^2(R)$ may be infinite even when I is a two-generated ideal. In connection with the result of Katzman, a natural question which arises in this study is whether $\operatorname{Ass}_R H_I^2(R)$ must be finite if R is integrally closed (and perhaps excellent). We give a positive answer to this question if the dimension of R is four (Corollary 2.8), although we do not know of a counterexample in any dimension. All known examples in which $\operatorname{Ass}_R H_I^2(R)$ is infinite and I is two-generated are in rings which are not integrally closed.

Our first main result (Theorem 2.4) shows if the cohomological dimension of I is at most 2, then the support of $H_I^2(M)$ is closed for all finite R-modules M. The cohomological dimension $\operatorname{cd}(I)$ of I is defined to be the supremum of the set of integers i such that $H_I^i(M) \neq 0$ for some R-module M. If $t = \operatorname{cd}(I)$ it can be easily shown that $H_I^t(R) \neq 0$. In particular, any ideal generated by two elements has cohomological dimension at most two, so that the support of $H_I^2(R)$ is closed if I is two generated. This follows quickly from a previous result of Hochster [8], as we describe later in this section.

We begin with a useful consequence of Gruson's Theorem (e.g., [25, Corollary 4.3]), which allows one to reduce to the case $M = R$ when considering the support of top local cohomology modules:

Proposition 2.1. *Let R be a Noetherian ring and I an ideal such that $\operatorname{cd}(I) = n$. Let M be a finitely generated R-module. Then*

$$\operatorname{Supp}_R H_I^n(M) = \operatorname{Supp}_R H_I^n(R/J),$$

where J is the annihilator of M. In particular, if $\operatorname{Supp}_R H_I^n(R/J)$ is closed then $\operatorname{Supp}_R H_I^n(M)$ is closed.

Proof. We may replace R by R/J to assume that M is faithful (note that the support of $H_I^n(M)$ is in any case contained in $V(J)$). This reduction does not change the local cohomologies, and furthermore $\operatorname{cd}_{R/J}((I+J)/J)$ is at most n, since any R/J-module is also an R-module. If $\operatorname{cd}_{R/J}((I+J)/J) < n$ there is nothing to prove as in this case $H_I^n(M) = 0$.

Because n is the cohomological dimension of I it follows that for every prime Q in R, there are isomorphisms,

$$H_I^n(M)_Q \cong \big(H_I^n(R) \otimes_R M\big)_Q \cong H_{I_Q}^n(R_Q) \otimes_{R_Q} M_Q.$$

Since M_Q is a faithful R_Q-module, Gruson's Theorem implies $H_{I_Q}^n(R_Q) \otimes_{R_Q} M_Q = 0$ if and only if $H_{I_Q}^n(R_Q) = 0$, which implies the proposition. $\square$

We will need the following easy lemma concerning exact sequences of flat modules:

Lemma 2.2. *Let S be a ring, and suppose that*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0$$

is an exact sequence of flat S-modules with $n \geqslant 1$. Suppose that I is an ideal and $IF_i = F_i$ for all $i \neq n$. Then $IF_n = F_n$.

Proof. We use induction on n. If $n = 1$, the result is immediate. If $n = 2$, we tensor the sequence with R/I, giving a short exact sequence,

$$0 \to F_2/IF_2 \to F_1/IF_1 \to F_0/IF_0 \to 0,$$

which proves the claim. If $n > 2$, truncate the sequence by letting K be the kernel of the map from F_1 to F_0. It follows that K is also flat, and by the case $n = 2$, $IK = K$. The result follows by the induction applied to the exact sequence $0 \to F_n \to F_{n-1} \to \cdots \to K \to 0$. $\square$

Let R be a ring and $\mathbf{x} = x_1, \ldots, x_n$ elements of R. The Čech complex of R with respect to $\mathbf{x}$ is defined by

$$\mathbf{C}(\mathbf{x}; R) := \bigotimes_{i=1}^{n} (0 \to R \to R_{x_i} \to 0).$$

This gives a complex of flat R-modules

$$0 \to R \to \bigoplus_i R_{x_i} \to \cdots \to R_{x_1 \cdots x_n} \to 0,$$

where R is in cohomological degree 0. See [3, Section 5.1] for basic properties of Čech complexes. For an R-module M the cohomology of $\mathbf{C}(\mathbf{x}; R) \otimes_R M$ is called the Čech cohomology of M with respect to $\mathbf{x}$. In the case R is Noetherian, the Čech cohomology is naturally isomorphic to the local cohomology of M with support in the ideal $(\mathbf{x})R$.

Corollary 2.3. *Let R be a ring, and let $\mathbf{x} = x_1, \ldots, x_n$ elements of R. Then $(\mathbf{x})R = R$ if and only if $\mathbf{C}(\mathbf{x}; R)$ is exact.*

Proof. Let $I = (\mathbf{x})R$. First assume that $\mathbf{C}(\mathbf{x}; R)$ is exact. Since $IR_f = R_f$ if $f \in I$, the conditions of Lemma 2.2 are met, and we can conclude that $I = R$. Conversely, if $I = R$, then the cohomology of $\mathbf{C}(\mathbf{x}; R)$ is isomorphic to the cohomology of $\mathbf{C}(1; R)$ (e.g., [3, Exercise 5.1.15]), which is exact. □

For an ideal I of a commutative ring R the ideal transform $D_I(R)$ of I defined by

$$D_I(R) := \varinjlim_n \operatorname{Hom}_R\bigl(I^n, R\bigr).$$

See [3, Section 2.2] for the basic properties of ideal transforms. We note that if Q is the total quotient ring of R and I contains a non-zerodivisor, there is a natural ring isomorphism

$$D_I(R) \to \bigcup_{n \geqslant 1} \bigl(R :_Q I^n\bigr).$$

For a finitely generated R-module M over a Noetherian ring R it is elementary to prove that $H_I^i(M) = 0$ for all i if and only if $IM = M$. Hence, $\operatorname{Supp}_R \bigoplus_{j \geqslant 0} H_I^j(M) = \operatorname{Supp}_R M/IM$, which is closed. (See also Section 5 of [1], which proves this in a much more general context.) Similarly, one has $\operatorname{Supp}_R \bigoplus_{j \geqslant 1} H_I^j(M) = \operatorname{Supp}_R M/(H_I^0(M) + IM)$, which is also closed. The following theorem extends these observations by one more degree in the case $M = R$:

Theorem 2.4. *Let R be a Noetherian ring and I an ideal. Then $\operatorname{Supp}_R \bigoplus_{j \geqslant 2} H_I^j(R)$ is closed. In particular, if M is a finitely generated R-module and $\operatorname{cd}(I) = 2$, then $\operatorname{Supp}_R H_I^2(M)$ is closed.*

Proof. Using Proposition 2.1, to prove the last claim of the theorem we may assume that $M = R$. Hence it suffices to prove the first assertion of the theorem. Since $H_I^j(R) \cong H_I^j(R/H_I^0(R))$ for $j > 0$, we may replace R by $R/H_I^0(R)$ to assume that $H_I^0(R) = 0$. Set $S = D_I(R)$, the ideal transform of I. We will prove that $\operatorname{Supp}_R \bigoplus_{j \geqslant 2} H_I^j(R) = V(IS \cap R)$, which will prove the theorem.

Because $H_I^0(R) = 0$, I contains a non-zerodivisor of R and S can be identified with a subring of the total quotient ring as above. Furthermore, $H_I^j(R) \cong H_I^j(S)$ for $j \geqslant 2$, and $H_I^1(S) = H_I^0(S) = 0$ [3, 2.2.8].

Let Q be a prime ideal of R. Then $H_I^j(R)_Q = 0$ for all $j \geqslant 2$ if and only if $H_I^j(S)_Q = 0$ for all $j \geqslant 0$ if and only if $IS_Q = S_Q$, using Corollary 2.3. This latter condition holds if and only if $IS \cap R \not\subset Q$, proving the theorem. □

As a consequence we get a proof of Question 1.1 in the case $n = 2$:

Corollary 2.5. *Let R be a Noetherian ring and I an ideal generated by 2 elements. Then $\operatorname{Supp}_R H_I^2(M)$ is closed for all finitely generated R-modules M.*

Using the ideas of the above proof we are also able to recover with a more conceptual proof a result of Hochster [8, Corollary 6.11] giving necessary and sufficient conditions for the vanishing of $H_I^2(R)$ where I is a two-generated ideal.

Proposition 2.6. *Let R be a Noetherian ring and $I = (x, y)$ an ideal of R. The following conditions are equivalent*:

(1) $H^2_I(R) = 0$.
(2) $x^n y^n \in (x^{n+1}, y^{n+1})$ *for some positive integer n.*

Proof. It suffices to prove that (2) implies (1). By taking a prime filtration for R, it suffices to show that $H^2_I(R/p) = 0$ for all $p \in \operatorname{Spec} R$. Hence, we may assume R is a domain. Let $S = D_I(R)$ be the ideal transform of I. As in the proof of Theorem 2.4, $H^2_I(R) = 0$ if and only if $IS = S$. By (2), $x^n y^n = r_1 x^{n+1} + r_2 y^{n+1}$ for some $r_1, r_2 \in R$. Hence in the field of fractions of R we have $1 = \frac{r_1}{y^n} x + \frac{r_2}{x^n} y$. We show that $\frac{r_1}{y^n}, \frac{r_2}{x^n} \in S$, giving $1 \in IS$ and completing the proof. But $\frac{r_1}{y^n} x^{n+1} = r_2 y - x^n \in R$. Hence, $\frac{r_1}{y^n}(x, y)^{2n} \subseteq R$ and $\frac{r_1}{y^n} \in S$. By symmetry, $\frac{r_2}{x^n} \in S$. □

Let x_1, x_2, y_1, y_2 be indeterminates over $\mathbb{Z}$. Hochster's result [8, Corollary 6.11] states that with

$$R_n = \mathbb{Z}[x_1, x_2, y_1, y_2]/\big(x_1^n x_2^n - y_1 x_1^{n+1} - y_2 x_2^{n+1}\big),$$

and $I = (x_1, x_2)$, $H^2_I(R_n) = 0$. This is easily seen to be equivalent to the proposition above.

To understand what is the real content of Proposition 2.6, consider $I = (x, y)$ and what it means for $H^2_I(R) = 0$. Computing this local cohomology with Čech cohomology, the vanishing means that for every $t \geqslant 1$, there must exist an n (depending on t) such that $x^n y^n \in (x^{n+t}, y^{n+t})$. The meaning of the proposition is that if such a containment holds for even $t = 1$ and some n, then this alone forces $H^2_I(R) = 0$. If something similar were true for larger numbers of generators, it would be easy to prove the support of the nth local cohomology of an ideal with n generators is always closed. However, this cannot be the case, due to an example of Paul Roberts [20]. He proved that if we set

$$R = \mathbb{Z}[x_1, x_2, x_3, y_1, y_2, y_3]/\big(x_1^2 x_2^2 x_3^2 - y_1 x_1^3 - y_2 x_2^3 - y_3 x_3^3\big),$$

and $I = (x_1, x_2, x_3)$, then $H^3_I(R) \neq 0$. It follows that for every $t \geqslant 1$ there exists a Noetherian ring R_t and $x_1, x_2, x_3 \in R_t$ such that $x_1^n x_2^n x_3^n \in (x_1^{n+t}, x_2^{n+t}, x_3^{n+t})$ and $H^3_{(x_1,x_2,x_3)}(R_t) \neq 0$. Specifically, let $t \geqslant 1$ and set $f_t = (x_1 x_2 x_3)^{2t} - y_1 x_1^{3t} - y_2 x_2^{3t} - y_3 x_3^{3t}$ and $R_t := \mathbb{Z}[x_1, x_2, x_3, y_1, y_2, y_3]/(f_t)$. Since R_t is free as a $\mathbb{Z}[x_1^t, x_2^t, x_3^t, y_1, y_2, y_3]/(f_t)$-module and the latter ring is isomorphic as a ring to R_1 via the map sending x_i to x_i^t for $i \in \{1, 2, 3\}$, we see that $H^3_{(x_1,x_2,x_3)}(R_t) \neq 0$. It is clear that $x_1^n x_2^n x_3^n \in (x_1^{n+t} x_2^{n+t} x_3^{n+t}) R_t$ for all $n \geqslant 2t$.

As a final application of the ideal transform, the following proposition shows that for I arbitrary, questions about the support or associated primes of $H^2_I(R)$ come down to related questions concerning the second local cohomology of a two-generated ideal.

Proposition 2.7. *Let R be a Noetherian ring and $I \subseteq R$ an ideal. Then there exists a two-generated ideal $J \subseteq I$ such that*:

(i) $\operatorname{Ass}_R H^2_J(R) = \operatorname{Ass}_R H^2_I(R) \cup [\operatorname{Ass}_R R/(J + H^0_I(R)) \cap D(I)]$, *where* $D(I) = \operatorname{Spec} R \setminus V(I)$.
(ii) $\operatorname{Supp}_R H^2_I(R) = \operatorname{Supp}_R \operatorname{Hom}_R(R/I, H^2_J(R))$.

In particular, $\operatorname{Ass}_R H_I^2(R)$ *is finite if and only if* $\operatorname{Ass}_R H_J^2(R)$ *is finite, and the support of* $H_I^2(R)$ *is closed if and only if the support of* $\operatorname{Hom}_R(R/I, H_J^2(R))$ *is closed.*

Proof. If I is nilpotent the result is trivial. We next show if statements (i) and (ii) hold for $R' = R/H_I^0(R)$ and $I' = IR'$ then they hold also for R and I. For, suppose (i) and (ii) hold for R' and I' and let $J' = (x, y)R' \subseteq I'$ be the ideal of R' whose existence is asserted. Clearly, we may assume $x, y \in I$. Let $J = (x, y)R$. By the change of rings principal and the fact that $H_J^0(H_I^0(R)) = H_I^0(R)$, we have $H_J^2(R) \cong H_{J'}^2(R')$ and $H_{I'}^2(R') \cong H_I^2(R)$. In particular, $H_J^2(R)$ is an R'-module and $\operatorname{Hom}_{R'}(R'/I', H_{J'}^2(R')) \cong \operatorname{Hom}_R(R/I, H_J^2(R))$. From this, it follows that statements (i) and (ii) hold for R and I.

We now assume $H_I^0(R) = 0$, i.e., that I contains a non-zerodivisor x. If we take $y \in I$ so that for any $P \in \operatorname{Ass}_R R/xR$, $y \in P$ if and only if $I \subseteq P$, then for $J := (x, y)R$, S is also the ideal transform of R with respect to J and x, y is a regular sequence (possibly improper) on S. (When R is a Krull domain, this is due to Nagata, see [18, p. 198]. For more general rings, see for example [11].) From the short exact sequence $0 \to S \xrightarrow{x} S \to S/xS \to 0$, we obtain the long exact sequence

$$0 \to H_I^1(S/xS) \to H_I^2(S) \xrightarrow{x} H_I^2(S) \to \cdots,$$

from which it follows that $\operatorname{Hom}_R(R/I, H_I^1(S/xS)) \cong \operatorname{Hom}_R(R/I, H_I^2(S))$. If we now repeat this argument on the long exact sequence in cohomology associated to the short exact sequence $0 \to S/xS \xrightarrow{y} S/xS \to S/JS \to 0$, we obtain

$$\operatorname{Hom}_R\bigl(R/I, H_I^2(S)\bigr) \cong \operatorname{Hom}_R(R/I, S/JS).$$

Applying the same argument to J shows that

$$\operatorname{Hom}_R\bigl(R/J, H_J^2(S)\bigr) \cong \operatorname{Hom}_R(R/J, S/JS) \cong S/JS.$$

Since S is the ideal transform of both I and J, we have $H_I^2(R) \cong H_I^2(S)$ and $H_J^2(R) \cong H_J^2(S)$ [3, 2.2.8]. Thus,

$$\begin{aligned}
\operatorname{Ass}_R H_I^2(R) &= \operatorname{Ass}_R \operatorname{Hom}_R\bigl(R/I, H_I^2(S)\bigr) \\
&= \operatorname{Ass}_R \operatorname{Hom}_R(R/I, S/JS) \\
&= V(I) \cap \operatorname{Ass}_R S/JS \\
&= V(I) \cap \operatorname{Ass}_R \operatorname{Hom}_R\bigl(R/J, H_J^2(S)\bigr) \\
&= V(I) \cap \operatorname{Ass}_R H_J^2(R).
\end{aligned}$$

Moreover, if $P \in \operatorname{Ass}_R H_J^2(R) = \operatorname{Ass}_R S/JS$ does not contain I, then $S_P = R_P$, so $P \in \operatorname{Ass}_R R/J$. The first statement in the proposition now follows. For the second statement, we note that from what we have just shown

$$\begin{aligned}\operatorname{Supp}_R H_I^2(R) &= \operatorname{Supp}_R \operatorname{Hom}\bigl(R/I, H_I^2(R)\bigr)\\ &= \operatorname{Supp}_R \operatorname{Hom}(R/I, S/JS)\\ &= \operatorname{Supp}_R \operatorname{Hom}_R\bigl(R/I, \operatorname{Hom}_R\bigl(R/J, H_J^2(S)\bigr)\bigr)\\ &= \operatorname{Supp}_R \operatorname{Hom}_R\bigl(R/I, H_J^2(R)\bigr),\end{aligned}$$

which is what we want. □

It is an open question whether there exists an ideal I in a four-dimensional local ring R such that for some i, $H_I^i(R)$ has infinitely many associated primes. (Swanson and Singh [23, Remark 4.2] provide an example of such a local cohomology module over a four-dimensional non-local ring.) It has been observed that if R is a four-dimensional local ring which is an isolated singularity or a UFD, then $H_I^i(R)$ has finitely many associated primes for all ideals I and $i \geqslant 0$ [17, Corollary 2.9]. We can improve these results (at least for excellent rings) using Proposition 2.7 as follows:

Corollary 2.8. *Let (R, m) be a normal, excellent local domain of dimension four. Then for any ideal $I \subseteq R$ and $i \geqslant 0$, $\operatorname{Ass}_R H_I^i(R)$ is finite.*

Proof. By [17], Proposition 1.1 and Corollary 2.4, we need only consider the case $i = 2$. By Proposition 2.7, there exists a two-generated ideal $J \subseteq I$ such that $\operatorname{Ass}_R H_I^2(R)$ is finite if $\operatorname{Ass}_R H_J^2(R)$ is finite. We now argue that $\operatorname{Ass}_R H_J^2(R)$ is finite.

Write $\sqrt{J} = L \cap K$, where L has pure height one and height $K \geqslant 2$. Since R is excellent and normal, for any prime ideal $P \subseteq R$, R_P is analytically irreducible. By the Hartshorne–Lichtenbaum vanishing theorem (see Theorem 8.2.1 in [3]), it follows that if height $P = 2$, then $H_L^2(R)_P = 0$.

Now, let $Q \in \operatorname{Ass}_R H_J^2(R)$. Then height $Q \geqslant 2$ and by the proof of Theorem 2.4 or by Proposition 2.7, $JS \cap R \subseteq Q$, where S denotes the ideal transform of R with respect to J. If $L \nsubseteq Q$, then $\sqrt{J_Q} = K_Q$. Thus, $Q \in \operatorname{Ass}_R H_K^2(R)$, a finite (possibly empty) set, since grade $K \geqslant 2$. If $K \nsubseteq Q$, then $\sqrt{J_Q} = L_Q$. By the preceding paragraph, we must have height $Q \geqslant 3$ and Q is either minimal over $JS \cap R$ or $Q = m$. Finally, if Q contains L and K, then Q contains $L + K$. Since height$(L + K) \geqslant 3$, we have that either Q is minimal over $L + K$ or $Q = m$. Thus, if $Q \in \operatorname{Ass}_R H_J^2(R)$, then either $Q \in \operatorname{Ass}_R H_K^2(R)$, Q is minimal over $JS \cap R$, Q is minimal over $L + K$, or $Q = m$. Therefore, $\operatorname{Ass}_R H_J^2(R)$ is finite and the proof is complete. □

3. The four-dimensional case

In this section we prove $\operatorname{Supp}_R H_I^i(M)$ is closed whenever R is local and $\dim R = 4$. The case when $\dim R \leqslant 3$ follows from [17, Corollary 2.6]. We first need a result concerning the existence of certain general elements. For ideals I and J of a ring R, we set

$$\bigl(J : I^{\infty}\bigr) := \bigcup_{n=1}^{\infty} \bigl(J : I^n\bigr).$$

Lemma 3.1. *Let R be a Noetherian ring and I an ideal of R. For any integer $n \geqslant 0$ there exists an n-generated ideal $J \subseteq I$ such that the following conditions hold*:

(1) $\operatorname{height}(J : I^\infty) \geqslant n$.
(2) $\operatorname{height}((J : I^\infty) + I) \geqslant n + 1$.
(3) $\sqrt{(J : I^\infty) \cap I} = \sqrt{J}$.

Proof. First we prove (3) for any ideal $J \subset I$. It suffices to prove that $\sqrt{(J : I^\infty) \cap I} \subset \sqrt{J}$ since the other containment is clear. If not there is a prime ideal P containing J, but not containing I or $(J : I^\infty)$. However, if P does not contain I, then $(J : I^\infty)$ is contained in P, a contradiction.

Next we prove that in general, $(J : I^\infty) + I$ is never contained in a minimal prime of $(J : I^\infty)$; as a consequence, (1) implies (2). If not, there is a minimal prime Q of $(J : I^\infty)$ which also contains I. It follows that in R_Q, $I_Q^N \subseteq (J : I^\infty)_Q$ for some integer N. But then some power of I_Q lies in J_Q, which means that $(J : I^\infty) \not\subset Q$, a contradiction.

Finally we prove (1) by induction on n. If $n = 0$ there is nothing to prove. Now assume the proposition holds for $n - 1$. Let $K = (x_1, \dots, x_{n-1})R$ be an ideal contained in I which satisfies (1) for $n - 1$. Choose $x_n \in I$ which is in no prime of height $n - 1$ containing $(K : I^\infty)$. This choice is possible since K satisfies (1) and (2) for $n - 1$. We claim that $J = K + (x_n)R$ satisfies condition (1) for n. If the height of $(J : I^\infty)$ is at most $n - 1$, then the height of the subideal $(K : I^\infty) + (x_n)R$ is also at most $n - 1$, contradicting the choice of x_n. □

This lemma enables us to prove the following:

Proposition 3.2. *Let R be a Noetherian ring, I an ideal of R, and M a finitely generated R-module, and n a positive integer. Suppose that $\operatorname{Supp}_R H_J^n(M)$ is closed for every n-generated ideal J contained in I. Then the set*

$$\left\{Q \in \operatorname{Min}_R H_I^n(M) \;\middle|\; \operatorname{ht} Q = n + 1\right\}$$

is finite.

Proof. Choose $x_1, \dots, x_n \in I$ as in Lemma 3.1, and let J be the ideal they generate. We claim that

$$\operatorname{Supp}_R H_I^n(M) \subset \operatorname{Supp}_R R/\big((J : I^\infty) + I\big) \cup \operatorname{Supp}_R H_J^n(M).$$

To see this claim, let $Q \in \operatorname{Supp}_R H_I^n(M)$. Clearly $I \subset Q$. If $Q \notin \operatorname{Supp}_R R/((J : I^\infty) + I)$, then necessarily $(J : I^\infty) \not\subset Q$, which forces $\sqrt{J_Q} = \sqrt{I_Q}$. In this case $H_{I_Q}^n(M_Q) = H_{J_Q}^n(M_Q)$, proving that $Q \in \operatorname{Supp}_R H_J^n(M)$.

Now suppose that $Q \in \operatorname{Min}_R H_I^n(M)$ has height $n + 1$. Note that by Lemma 3.1, the height of $(J : I^\infty) + I$ is at least $n + 1$, so that the set of primes of height $n + 1$ which contain $(J : I^\infty) + I$ is finite. Assuming that Q is not one of these primes, it follows from the above claim that $Q \in \operatorname{Supp}_R H_J^n(M)$, and moreover that $\sqrt{J_Q} = \sqrt{I_Q}$. Since $Q \in \operatorname{Min}_R H_I^n(M)$, it follows that $Q \in \operatorname{Min}_R H_J^n(M)$, which is a finite set by assumption. □

The following corollary follows immediately from Proposition 3.2 and Theorem 2.4.

Corollary 3.3. *Let R be a Noetherian ring, I an ideal of R, and M a finitely generated R-module. Then the set $\{Q \in \operatorname{Min}_R H_I^2(M) \mid \operatorname{ht} Q = 3\}$ is finite.*

We now prove that local cohomology modules over local rings of dimension four have closed support.

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R \leqslant 4$, I an ideal of R, and M a finitely generated R-module. Then $\operatorname{Supp}_R H_I^i(M)$ is closed for all $i \geqslant 0$.*

Proof. By Proposition 1.1 and Corollary 2.4 of [17], it is enough to consider the support of $H_I^2(M)$ when $\dim R = 4$. Let Q be minimal in $\operatorname{Supp}_R H_I^2(M)$. If $Q = \mathfrak{m}$, we are done. Otherwise, Q has height two or three. By [17, Proposition 2.2], there are only finitely many primes of height two in $\operatorname{Supp}_R H_I^2(M)$. By Corollary 2.5, $\operatorname{Supp}_R H_J^2(M)$ is closed for all two-generated ideals J of R. Hence we may apply Corollary 3.3 to conclude that there are at most finitely many primes of height three which are minimal in $\operatorname{Supp}_R H_I^2(M)$. $\square$

4. A result in characteristic *p*

Using the action of Frobenius on local cohomology, G. Lyubeznik proved that if R is a Noetherian ring of characteristic $p > 0$ and I is an ideal generated by n elements, then $\operatorname{Supp}_R H_I^n(R)$ is closed (cf. [13, Theorem 2.10]). The following gives a generalization of this result:

Theorem 4.1. *Let R be a Noetherian ring of prime characteristic p and $I = (\mathbf{x})R$ an ideal of R, where $\mathbf{x} = x_1, \dots, x_n$. Let $H^i(\mathbf{x}; R)$ denote the ith Koszul cohomology of $\mathbf{x}$ on R. Suppose $\operatorname{pd}_R H^i(\mathbf{x}; R) < \infty$ for $i > j$. Then $\operatorname{Supp}_R H_I^i(R)$ is closed for all $i \geqslant j$.*

Proof. Let $\mathbf{K}$ denote the Koszul cochain complex with respect to $\mathbf{x}$ on R and let $\phi^i : K^i \to K^{i+1}$ be the ith differential. Let F denote the Frobenius functor from the category of R-modules to itself (e.g., [19, I.1.2]). For $q = p^e$ we set $\mathbf{K}[q] := F^e(\mathbf{K}) \cong \mathbf{K}(\mathbf{x}^q; R)$, $\phi^i[q] := F^e(\phi^i)$, $Z[q]^i := \ker \phi^i[q]$, and $B[q]^i := \operatorname{im} \phi^{i-1}[q]$. We write Z^i and B^i for $Z[1]^i$ and $B[1]^i$, respectively. If M is a submodule of R^n for some n, we let $M^{[q]}$ denote the submodule of R^n generated by all elements of the form $\alpha^q = (a_1^q, \dots, a_n^q)$ where $\alpha = (a_1, \dots, a_n) \in M$. (This is easily seen to be independent of the choice of basis for R^n.)

Claim 1. *For $i \geqslant j$, $Z[q]^i = (Z^i)^{[q]}$ for all q.*

Proof. Suppose $i \geqslant j$. From the hypothesis and the short exact sequences

$$0 \to Z^i/B^i \to K^i/B^i \to B^{i+1} \to 0$$

and

$$0 \to B^{i+1} \to K^{i+1} \to K^{i+1}/B^{i+1} \to 0$$

we see that $\operatorname{pd}_R K^i/B^i < \infty$ for $i > j$. Suppose $i \geqslant j$ and let

$$0 \to G_r \to \cdots \to G_2 \xrightarrow{\psi^i} K^i \xrightarrow{\phi^i} K^{i+1} \to K^{i+1}/B^{i+1} \to 0$$

be a finite free resolution of K^{i+1}/B^{i+1}. Applying Frobenius we get

$$G_2 \xrightarrow{\psi^i[q]} K^i \xrightarrow{\phi^i[q]} K^{i+1}$$

is exact for all q (see [4, Theorem 8.2.7], for example). Hence, $Z[q]^i = \operatorname{im}\psi^i[q] = (\operatorname{im}\psi^i)^{[q]} = (Z^i)^{[q]}$. This proves Claim 1. $\square$

Now, $H_I^i(R) = \varinjlim H^i(\mathbf{x}^q; R) = H^i(\varinjlim \mathbf{K}[q])$ where the direct limit is taken over all $q = p^e$. For pth powers $q \leqslant q'$ let $f_{q,q'} : \mathbf{K}[q] \to \mathbf{K}[q']$ be the chain map in the direct system $\{\mathbf{K}[q] \mid q = p^e,\ e \geqslant 0\}$. For convenience, denote $f_{1,q}$ by f_q. It is easily checked that $f_{q',qq'} = f_q[q']$ for any q, q'. Following standard notation, we let $(f_q^i)^* : H^i(\mathbf{K}) \to H^i(\mathbf{K}[q])$ denote the map on cohomology induced by f_q^i.

Claim 2. *Let $i \geqslant j$ be given. If $(f_q^i)^* = 0$ for some pth power q then $(f_{q',qq'}^i)^* = 0$ for all pth powers q'.*

Proof. By the remarks above, it is enough to show $(f_q[q']^i)^* = 0$ for any $q' = p^{e'}$. Applying $F^{e'}$ to the diagram

$$\begin{array}{ccccc} K^{i-1} & \xrightarrow[\phi^{i-1}]{} & K^i & \xrightarrow[\phi^i]{} & K^{i+1} \\ \downarrow{\scriptstyle f_q^{i-1}} & & \downarrow{\scriptstyle f_q^i} & & \downarrow{\scriptstyle f_q^{i+1}} \\ K[q]^{i-1} & \xrightarrow[\phi^{i-1}[q]]{} & K[q]^i & \xrightarrow[\phi^i[q]]{} & K[q]^{i+1} \end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccccc} K[q']^{i-1} & \xrightarrow[\phi^{i-1}[q']]{} & K[q']^i & \xrightarrow[\phi^i[q']]{} & K[q']^{i+1} \\ \downarrow{\scriptstyle f_q^{i-1}[q']} & & \downarrow{\scriptstyle f_q^i[q']} & & \downarrow{\scriptstyle f_q^{i+1}[q']} \\ K[qq']^{i-1} & \xrightarrow[\phi^{i-1}[qq']]{} & K[qq']^i & \xrightarrow[\phi^i[qq']]{} & K[qq']^{i+1}. \end{array}$$

Since $Z^i[q'] = (Z^i)^{[q']}$ (Claim 1), it is enough to show that for $\alpha \in Z^i$, $f_q^i[q'](\alpha^{q'}) \in B^i[qq']$. Since $(f_q^i)^* = 0$, $f_q^i(\alpha) \in B^i[q]$. But $f_q^i[q'](\alpha^{q'}) = f_q^i(\alpha)^{q'} \in (B^i[q])^{[q']} = B^i[qq']$. This proves Claim 2. $\square$

Thus for $i \geqslant j$, $H_I^i(R) = 0$ if and only if for some q the map $(f_q^i)^* : H^i(\mathbf{x}; R) \to H^i(\mathbf{x}^q; R)$ is zero. Let A_q^i be the kernel of $(f_q^i)^*$ and A^i the union of A_q^i over all q. Then $H_I^i(R) = 0$ if and only if $A^i = H^i(\mathbf{x}; R)$. Everything localizes (including the hypothesis), so $\operatorname{Supp}_R H_I^i(R) = \operatorname{Supp}_R H^i(\mathbf{x}; R)/A^i$, which is closed. $\square$

Corollary 4.2. *Let R be a Noetherian ring of prime characteristic p and I an ideal generated by n elements. Then $\operatorname{Supp}_R H_I^n(M)$ is closed for all finitely generated R-modules M.*

Proof. Combine Lyubeznik's result (the case $j = n$ in Theorem 4.1) with Proposition 2.1. □

5. Some miscellaneous reductions

The following proposition is a reduction result which use ideas of M. Hellus [7]. Only one additional twist (using a weak version of the Dedekind–Mertens Lemma) is needed to complete the argument.

Proposition 5.1. *Let R be a Noetherian ring and $I = (x_1, \ldots, x_{n+k})R$ where $k \geqslant 0$, and $n \geqslant k+4$. Let M be an R-module. Then there exists an ideal $J = (y_1, \ldots, y_{n+k-1})R \subseteq I$ such that $H_I^n(M) \cong H_J^{n-1}(M)$. Furthermore*:

(a) *If* $\operatorname{depth}_I R \geqslant 2$ *then one can choose the elements y_j so that* $\operatorname{depth}_J R \geqslant 2$.
(b) *If R is $\mathbb{Z}$-graded and each x_i is homogeneous of positive degree, then one can choose each y_j to be homogeneous of positive degree.*

Proof. Set s equal to the floor of $(n+k)/2$. Let $I_1 = (x_1, \ldots, x_s)$ and $I_2 = (x_{s+1}, \ldots, x_{n+k})$, so that $I_1 + I_2 = I$. The Mayer–Vietoris sequence for local cohomology gives a long exact sequence,

$$\cdots \to H_{I_1}^{n-1}(M) \oplus H_{I_2}^{n-1}(M) \to H_{I_1 \cap I_2}^{n-1}(M) \to H_I^n(M) \to H_{I_1}^n(M) \oplus H_{I_2}^n(M) \to \cdots.$$

Since $n \geqslant k+4$, $n-1 > n+k-s$ and $n-1 > s$. It follows that there is an isomorphism

$$H_{I_1 \cap I_2}^{n-1}(M) \cong H_I^n(M).$$

We will be done if we can prove that $I_1 \cap I_2$ is generated up to radical by $n+k-1$ elements. However, up to radical this ideal is the same as the product ideal $I_1 I_2$, and this ideal is generated up to radical by $s + (n+k-s) - 1 = n+k-1$ elements [24, Corollary 1.7.6] which can clearly be chosen in I. Moreover, if R is $\mathbb{Z}$-graded and $\deg x_i > 0$ for all i, then one can replace each x_i by a positive power so that the x_i all have the same positive degree. By the proof of [24, Corollary 1.7.6], the elements y_j can be chosen to be homogeneous of positive degree (and in I) as well. Finally, if $\operatorname{depth}_I R \geqslant 2$ then, since $s \geqslant 2$, one can replace the generators x_i, if necessary, so that I_1 and I_2 (and hence J) each have grade at least two. If the original generators of I are homogeneous of positive degree, one can do this replacement so that the new generators also are homogeneous of positive degree. □

Corollary 5.2. *Let R be a Noetherian ring and $I = (x_1, \ldots, x_n)R$ where $n \geqslant 4$. Let M be an R-module. Then there exists an ideal $J = (y_1, \ldots, y_{n-1})R \subseteq I$ such that $H_I^n(M) \cong H_J^{n-1}(M)$. Furthermore*:

(a) *If* $\operatorname{depth}_I R \geqslant 2$ *then one can choose the elements y_j so that* $\operatorname{depth}_J R \geqslant 2$.
(b) *If R is $\mathbb{Z}$-graded and each x_i is homogeneous of positive degree, then one can choose each y_j to be homogeneous of positive degree.*

Proof. Let $k = 0$ in Proposition 5.1. □

In connection with Question 1.1, we have the following:

Corollary 5.3. *Let R be a Noetherian ring and M a finitely generated R-module. The following conditions are equivalent*:

(a) *For all positive integers n,* $\operatorname{Supp}_R H_I^n(M)$ *is closed for every n-generated ideal I of R.*
(b) $\operatorname{Supp}_R H_I^3(M)$ *is closed for every three-generated ideal I of R.*

Proof. It suffices to prove that (b) implies (a). Corollary 5.2 shows that if $n \geqslant 3$ then $\operatorname{Supp}_R H_I^n(M)$ is closed for every n-generated ideal I of R. The case $n = 2$ is handled by Corollary 2.5, while the cases $n = 0$ and $n = 1$ are trivial or well known. □

We also make explicit the following result for $(n + 1)$-generated ideals, in order to apply it to Cohen–Macaulay rings.

Corollary 5.4. *Let R be a Noetherian ring and $I = (x_1, \dots, x_{n+1})R$ an ideal. If $n \geqslant 5$ then there exists an ideal $J = (y_1, \dots, y_n)R \subseteq I$ such that $H_I^n(R) \cong H_J^{n-1}(R)$.*

Proof. Let $k = 1$ in Proposition 5.1. □

Consequently, we have:

Corollary 5.5. *Let R be a Noetherian ring. The following conditions are equivalent*:

(a) *For all positive integers n,* $\operatorname{Supp}_R H_I^n(R)$ *is closed for every $(n + 1)$-generated ideal I.*
(b) $\operatorname{Supp}_R H_I^2(R)$ *is closed for every three-generated ideal I of R,* $\operatorname{Supp}_R H_I^3(R)$ *is closed for every four-generated ideal I of R, and* $\operatorname{Supp}_R H_I^4(R)$ *is closed for every five-generated ideal I.*

Proof. The above corollary shows that if $n \geqslant 5$, we can reduce the problem to proving the support is closed for $\operatorname{Supp}_R H_I^4(R)$ for a five-generated ideal. If $n = 1$, the support is always closed. □

We can do much better in a Cohen–Macaulay local ring, again following Hellus's work. Namely:

Corollary 5.6. *Let R be a Cohen–Macaulay Noetherian local ring. The following conditions are equivalent*:

(a) *For all positive integers n,* $\operatorname{Supp}_R H_I^n(R)$ *is closed for every ideal I.*
(b) $\operatorname{Supp}_R H_I^2(R)$ *is closed for every three-generated ideal I of R,* $\operatorname{Supp}_R H_I^3(R)$ *is closed for every four-generated ideal I of R, and* $\operatorname{Supp}_R H_I^4(R)$ *is closed for every five-generated ideal I.*

Proof. It suffices to prove that $H_I^n(R) \cong H_J^n(R)$ for some ideal $J \subset I$ such that J is generated by $n + 1$ elements. Then we can apply Corollary 5.5 to finish the proof. Choose $J \subset I$ as in Lemma 3.1 for the value $n+1$. Set $K = (J : I^\infty)$. The Mayer–Vietoris sequence on local cohomology gives an exact sequence

$$\cdots \to H^n_{I+K}(R) \to H^n_I(R) \oplus H^n_K(R) \to H^n_{I\cap K}(R) \to H^{n+1}_{I+K}(R) \to \cdots.$$

By Lemma 3.1, the height of $I+K$ is at least $n+2$, and the height of K is at least $n+1$. Since R is Cohen–Macaulay, this implies that the above exact sequence collapses to an isomorphism,

$$H^n_I(R) \cong H^n_{I\cap K}(R).$$

However, by Lemma 3.1, $I \cap K$ is the same as J up to radical. $\square$

Remark 5.7. One can omit the hypothesis that $\operatorname{Supp}_R H^2_I(R)$ is closed for every three-generated ideal I of R if one assumes, e.g., that R is a UFD. In fact if R is a Cohen–Macaulay UFD, then $\operatorname{Supp}_R H^2_I(R)$ has only finitely many associated primes for any ideal I. This follows by using the Mayer–Vietoris sequence to reduce to the case I has height two, for which the result holds since R is Cohen–Macaulay.

6. A result in equicharacteristic zero

Let A be an $m \times n$ matrix with entries in a commutative ring R. For $1 \leqslant k \leqslant \min\{m,n\}$ we let $I_k(A)$ denote the ideal of R generated by the $k \times k$ minors of A. In the case A is a 2×3 matrix and R is a ring of prime characteristic it is well known that $H^3_{I_2(A)}(R)$ vanishes, since it vanishes when A is a generic matrix over a field of characteristic p [19, III.4.1]. In the case A is a generic 2×3 matrix over a field of characteristic zero, a result of Hochster says that $H^3_{I_2(A)}(R) \neq 0$. The following theorem sheds additional light on the vanishing (or non-vanishing) of $H^3_{I_2(A)}(R)$ in the case A is an arbitrary 2×3 matrix with entries in a ring of equicharacteristic zero.

Theorem 6.1. *Let R be a Noetherian ring containing a field k of characteristic zero and $A = (a_{ij})$ a 2×3 matrix with entries from R. Then for any R-module M*

$$H^3_{I_2(A)}(M) \cong H^6_{I_1(A)}(M).$$

Moreover, if R is a Q-graded ring for some abelian subgroup $Q \subseteq \mathbb{Z}^d$ such that the entries and the 2×2 minors of A are homogeneous and M is a graded R-module, then the isomorphism above is a degree zero homogeneous map.

Proof. Let $S = k[t_{ij} \mid 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 3]$ be a polynomial ring in six variables over k and T the 2×3 matrix (t_{ij}). We consider R as an S-algebra via the ring homomorphism $\phi : S \to R$ which fixes k and maps t_{ij} to a_{ij}. If R is Q-graded, we give S a Q-grading by letting $\deg t_{ij} = \deg a_{ij}$. By the change of rings principal we have that $H^6_{I_1(A)}(M) \cong H^6_{I_1(T)}(S) \otimes_S M$ and $H^3_{I_2(A)}(M) \cong H^3_{I_2(T)}(S) \otimes_S M$. Moreover, these are degree zero isomorphisms in the graded case. Hence, it is enough to prove the result in the case $R = M = S$ and $A = T$.

Let $I = I_1(T)$ and $J = I_2(T)$. To prove the graded case simultaneously, we assume S is endowed with a Q-grading for some abelian subgroup $Q \subseteq \mathbb{Z}^d$ such that the entries and the minors of T are homogeneous; e.g., $Q = \mathbb{Z}$ and $\deg t_{ij} = 1$ for all i, j. It is well known that

$H^3_J(S)$ is non-zero. By the work of Lyubeznik [15], this module is actually Artinian and injective. In fact, $H^3_J(S) \cong E$, where E is the injective hull of the field S/I [26, Example 6.1]. Furthermore, as $H^3_J(S)$ is graded, we must have $H^3_J(S) \cong E(g)$ as graded Q-modules for some $g \in Q$. At the same time, $H^6_I(S)$ is graded and also isomorphic to E (e.g., [4, Theorem 3.6.19]). Hence, there is a graded isomorphism $H^6_I(S) \cong E(h)$ for some $h \in Q$. To show that $h = g$, it is enough to show that the socle generators of $H^3_J(S)$ and $H^6_I(S)$ have the same degree. It is well known that the socle of $H^6_I(S)$ is generated by the image of $\prod_{ij} t_{ij}^{-1}$ in $H^6(\mathbf{C}(\mathbf{t}; S))$, where $\mathbf{C}(\mathbf{t}; S)$ is the Čech complex on the t_{ij}. Hence, the socle degree of $H^6_J(S)$ is $-\sum_{ij} \deg t_{ij}$. Let B be the graded subring of S generated over k by the 2×2 minors Δ_1, Δ_2, and Δ_3 of T. It is easily seen that Δ_1, Δ_2, and Δ_3 are algebraically independent over k and thus the socle of $H^3_{(\Delta_1,\Delta_2,\Delta_3)B}(B)$ is generated by the image of $\frac{1}{\Delta_1\Delta_2\Delta_3}$ in $H^3(\mathbf{C}(\boldsymbol{\Delta}; B))$. By the Hochster and Roberts Theorem [9], the inclusion $B \to S$ splits as graded B-modules. (Note that B is the ring of invariants of $\mathrm{SL}_2(k)$ acting on S.) Hence, the graded map $H^3_{(\Delta_1,\Delta_2,\Delta_3)B}(B) \to H^3_J(S)$ is an injection. In particular, the image of the element $\frac{1}{\Delta_1\Delta_2\Delta_3}$ in $H^3_J(S)$, which is easily seen to be annihilated by $(t_{ij})S$, is non-zero. By our assumptions on the grading of S, we have $\deg \frac{1}{\Delta_1\Delta_2\Delta_3} = -(\deg \Delta_1 + \deg \Delta_2 + \deg \Delta_3) = -\sum_{ij} \deg t_{ij}$, completing the proof. □

Remark 6.2. We note that Theorem 6.1 holds for an arbitrary commutative ring containing $\mathbb{Q}$ if local cohomology is replaced by Čech cohomology.

We will need the following lemma in the proof of Corollary 6.4.

Lemma 6.3. *Let (R, m) be a Noetherian local ring and A a 2×3 matrix with entries from R. Suppose* $\operatorname{depth}_{I_1(A)} R \geqslant 2$. *Then there exists a 2×3 matrix B such that $I_1(B) = I_1(A)$ and* $\operatorname{depth}_{I_2(B)} R \geqslant 2$.

Proof. We first claim that if $Q_1, \ldots, Q_n$ are prime but not maximal ideals, then there exist an infinite sequence of elements r_i, $1 \leqslant i < \infty$, such that for all $i \neq j$, $r_i - r_j \notin Q_t$ for all $1 \leqslant t \leqslant n$. We prove this by induction on n. The case $n = 1$ is clear since R/Q_1 must be infinite. For $n > 1$, reorder the Q_t if necessary so that Q_1 is minimal among the set of Q_j. Choose $c \in Q_2 \cap \cdots \cap Q_n$ such that $c \notin Q_1$. By induction there exist an infinite sequence of elements r_i, $1 \leqslant i < \infty$, such that for all $i \neq j$, $r_i - r_j \notin Q_t$ for all $2 \leqslant t \leqslant n$. If infinitely many of these r_i have distinct cosets in R/Q_1, choose this subsequence. If not, infinitely many represent the same coset in R/Q_1, so by choosing a subsequence and reindexing, we can assume $r_i + Q_1 = r_j + Q_1$ for all i, j. Now replace r_i by $c^i + r_i$. These elements represent distinct cosets modulo Q_1 and $r_i + c^i + Q_t = r_i + Q_t$ for all $2 \leqslant t \leqslant n$. This proves the claim. Now let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

By standard arguments, we can assume that any two of the elements $\{a, b, c, d, e, f\}$ form a regular sequence. For $r \in R$ let

$$A_r := \begin{pmatrix} a & b \\ d & e + ra \end{pmatrix}.$$

Let Δ_r denote the determinant of A_r. We assert that we can choose r so that Δ_r is a non-zerodivisor. Let $P_1, \dots, P_r$ be the associated primes of R and suppose that Δ_r is a zero-divisor for all $r \in R$. Choose infinitely many r_i for $P_1, \dots, P_r$ as in our claim. There exists one of the primes P_j which contains infinitely many of the Δ_{r_i}. Since $\Delta_{r_i} - \Delta_{r_j} = (r_i - r_j)a^2$, this implies that $a \in P_j$, a contradiction. Thus we may assume that $ae - bd$ is a non-zerodivisor. Furthermore note that a, b and a, d have not changed and are therefore regular sequences.

Let $Q_1, \dots, Q_t$ be the associated primes of $R/(ae - db)$, and choose r_i for these primes as in the claim. Set

$$B_i := \begin{pmatrix} a & c \\ d & f + r_i a \end{pmatrix}.$$

Put $\delta_i = \det(B_i)$. We claim that for suitable i, $ae - bd, \delta_i$ form a regular sequence. If not, then there exists a Q_j which contains infinitely many of the δ_i. But then Q_j contains $\delta_i - \delta_k = (r_i - r_k)a^2$ for infinitely many $i \neq k$, which implies that $a \in Q_j$. Since $ae - bd \in Q_j$ it follows that either b or d is also in Q_j, a contradiction since the $\operatorname{depth}_{Q_j}(R) = 1$. $\square$

The following corollary reduces Question 1.1 in the case $n \geqslant 6$ and the ring contains a field of characteristic zero case to the case $n = 3$ and I is the ideal of maximal minors of a 2×3 matrix:

Corollary 6.4. *Let R be a Noetherian ring containing a field of characteristic zero. Let $I = (x_1, \dots, x_n)R$ where $n \geqslant 6$. Then there exists a 2×3 matrix A with entries from R such that $H_I^n(R) \cong H^3_{I_2(A)}(R)$. Moreover:*

(a) *If R is local and* $\operatorname{depth}_I R \geqslant 2$*, then A can be chosen such that* $\operatorname{depth}_{I_2(A)} R \geqslant 2$*, and in this case $I_2(A)$ has projective dimension one.*
(b) *If R is $\mathbb{Z}$-graded and each x_i is homogeneous of positive degree, then A can be chosen so that all its entries are homogeneous of positive degree and the isomorphism above is homogeneous of degree zero.*

Proof. Combine Corollary 5.2, Theorem 6.1, and Lemma 6.3. $\square$

Corollary 6.5. *Let R be a Noetherian ring of dimension at most five and A a 2×3 matrix with entries from R. Assume that R contains a field of characteristic zero. Then $H^3_{I_2(A)}(R) = 0$.*

Proof. Since $6 > \dim R$, $H^6_{I_1(A)}(R) = 0$. $\square$

Example 6.6. The result of Corollary 6.5 gives some interesting new information concerning the famous example of Hartshorne [6], namely in $S = k[x, y, u, v]$ that the ideal $I = (x, y) \cap (u, v)$ cannot be set-theoretically generated by two elements. This follows from the Mayer–Vietoris sequence applied to the ideal (x, y) and (u, v), which forces $H^3_I(S) \neq 0$. If k has characteristic p, then it is easy to see that I cannot even be set-theoretically Cohen–Macaulay, i.e., there does not exist an ideal $J \subset I$ such that S/J is Cohen–Macaulay and such that $\sqrt{J} = I$. This follows since in characteristic p, $H^3_J(S) = 0$. Corollary 6.5 gives that even in characteristic zero, there does not exist an unmixed ideal $J \subset I$ with $\sqrt{J} = I$ such that J has three generators. For by the syzygy theorem [5], S/J is necessarily Cohen–Macaulay, and then by Corollary 6.5, it follows that $H^3_J(S) = 0$.

This also implies that if we intersect I with an ideal generated by a regular sequence f, g, such that (f, g) is not contained in either (x, y) nor (u, v), then the new ideal $K = I \cap (f, g)$ cannot be generated up to radical by two elements a, b such that $(a, b)_P = (f, g)_P$ for all minimal primes P of (f, g). For if so, then the link $J := ((a, b) : (f, g))$ is contained in I, $\sqrt{J} = I$, R/J is Cohen–Macaulay, and J is at most three-generated (see [4, Corollary 2.3.10]), contradicting the work above.

The condition on the equality of (a, b) and (f, g) after localization at minimal primes of (f, g) is necessary. For example, if we intersect I with (x, u), then the resulting ideal (xv, xu, yu) is generated up to radical by two elements, e.g., by $(xu, xv^2 + uy^2)$.

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