FITTING IDEALS AND FINITE PROJECTIVE DIMENSION

C. Huneke, D. Jorgensen and D. Katz

September 10, 2003

Introduction

Throughout we let $(T, \mathfrak{m}, k)$ denote a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. We let $I \subseteq T$ be an ideal generated by a regular sequence of length c and set $R := T/I$. In the important paper [A], Avramov addresses the following question. Given a finitely generated R-module M, when does M have finite projective dimension over a ring of the form T/J, where J is generated by part (or all) of a set of minimal generators for I? The paper [A] gives a fairly complete answer to this question that is expressed in terms of the geometry of varieties in affine space defined by annihilators of certain graded modules derived from resolutions over R. In an attempt to understand these ideas more fully, we became interested in the idea that one might answer the question at hand by using data about M (or its syzygies) coming from T, in particular, information gleaned from various Fitting ideals defined over T. The following theorem from section two is one of our main results. We use $\mathrm{Fitt}_T(M)$ to denote the Fitting ideal of M.

Theorem 2.2. *Suppose M has a rank over R. Then M is free over R if and only if $\mathrm{Fitt}_T(M)$ is grade unmixed.*

Recall that an ideal in a Noetherian ring is grade unmixed if all of its associated primes have the same grade. The theorem has two immediate consequences if the ring T satisfies Serre's condition S_{c+1}. The first is the suprising fact that the Fitting ideal of *any* non-free R-module with a rank has embedded components, and the second is that M has finite projective dimension over R if and only the Fitting ideals over T of sufficiently high syzygies of M over R are unmixed.

The first author was partially supported by NSF grant DMS-0098654.
The second author was partially supported by a grant from the NSA.
The third author was partially supported by the University of Kansas General Research Fund.

Carrying the theme of Fitting ideals determining finite projective dimension further, in the third section we show how the Fitting ideal of a high syzygy of M determines whether or not M has finite projective dimension over intermediate complete intersections of codimension $c-1$. The relevance of this comes from the fact that if k is algebraically closed, M has finite projective dimension over R if and only if it has finite projective dimension over every intermediate complete intersection of codimension $c-1$. Thus, if M does not have finite projective dimension over R, it becomes of interest to determine over which, if any, intermediate complete intersections M has finite projective dimension. In section three, we show that, at least in codimension $c-1$, again the answer is determined by Fitting ideals.

Section 1

In this section we fix our notation and provide a few preliminary results, including an elementary presentation of a generalization of the result implicit in [A] that finite projective dimension over R is determined in codimension one. We assume throughout that $c \geq 2$. We will often assume that M has a positive rank. This means that for some $r > 0$ and all $P \in \text{Ass}(R)$, M_P is a free R_P-module of rank r. We fix $d := \text{depth}(R)$ and $\delta := \text{depth}(M)$ and assume throughout that $\delta \leq d$. We also fix a minimal presentation

$$T^m \xrightarrow{\phi} T^n \to M \to 0,$$

so that the Fitting ideal of M is the ideal generated by the $n \times n$ minors of the map ϕ with respect to some (equivalently an arbitrary) choice of bases for the two free modules. We denote this ideal by $\text{Fitt}_T(M)$, or by $\text{Fitt}(M)$ when the ring is clear. In fact, $\text{Fitt}_T(M)$ is independent of the presentation of M and respects change of rings. In particular, if T maps onto the ring S and M is also an S-module, then $\text{Fitt}_S(M) = I_n(\phi)S$. By $\Omega^i_R(M)$ we denote an i^{th} syzygy of M over R. The Fitting ideals of all choices of i^{th} syzygies are the same. Finally, by the term 'intermediate complete intersection of codimension t' we mean a ring of the form T/J, where J is generated by a set of t minimal generators for I and $1 \leq t \leq c-1$.

Now we turn to a proof of the fact that M has finite projective dimension over R if and only if M has finite projective dimension over every intermediate complete intersection of codimension one. We first require a lemma, interesting in its own right.

1.1 Lemma. *Let T, R be as above and let M be any finitely generated R-module. Assume $c = 2$, $h \in I \backslash mI$ and set $S := T/hT$. Assume that $\text{pd}_S(M) = 1$ and that M has no free summands over R and write $I := (f, h)T$, $Z := \Omega^1_T(M)$ with $Z \subseteq T^n$. Then, writing "$^-$" to denote images in $Z \otimes_T k$, $\overline{f \cdot e_j}$ belongs to the span of $\{\overline{h \cdot e_1}, \dots, \overline{h \cdot e_n}\}$, for all $1 \leq j \leq n$. Moreover, let A be the $n \times n$ transition* *matrix over k whose columns consist of the coefficients resulting from expressing the $\overline{f\cdot e_j}$ in terms of the $\overline{h\cdot e_i}$ and for g a minimal generator of I, set $S_g := T/gT$. Then the following are equivalent :*

(1) $\mathrm{pd}_{S_g}(M) < \infty$ *(i.e.,* $\mathrm{pd}_{S_g}(M) = 1$*).*
(2) $\{\overline{g\cdot e_j}\}$, $1 \le j \le n$, *are linearly independent in* $Z \otimes_T k$*.*
(3) $\mu(\Omega^1_{S_g}(M)) = n$*.*

Furthermore, if for some unit λ, g is equivalent to $f - \lambda h$ modulo $\mathfrak{m} I$, (1)-(3) are equivalent to

(4) $A - \lambda I_n$ *has maximal rank, i.e.,* λ *is not an eigenvalue of* A*.*

Proof. Let ψ be an $n \times n$ matrix with entries in $\mathfrak{m}$ such that

$$0 \longrightarrow S^n \xrightarrow{\psi\otimes 1_S} S^n \longrightarrow M \longrightarrow 0$$

is exact. Then it follows readily from the Buchsbaum-Eisenbud exactness criteria that

$$0 \longrightarrow T^n \xrightarrow{\begin{pmatrix} -h\cdot I_n \\ \psi \end{pmatrix}} T^{2n} \xrightarrow{(\psi \quad h\cdot I_n)} T^n \longrightarrow M \longrightarrow 0$$

is also exact. Therefore $\mu(Z) = 2n$ and Z is spanned by the columns of ψ and $h\cdot I_n$. Now suppose some $\overline{f\cdot e_j}$ does not belong to the span of the $\overline{h\cdot e_i}$, say $\overline{f\cdot e_1}$. Then there exists an $n\times n$ matrix $\psi' = (f\cdot e_1 \ *)$ such that Z is spanned by the columns of ψ' and $h\cdot I_n$. Therefore, since $\mathrm{Fitt}_T(M)$ has grade two,

$$0 \longrightarrow S^n \xrightarrow{\psi'\otimes 1_S} S^n \longrightarrow M \longrightarrow 0$$

is exact. If we tensor with R, we get a short exact sequence

$$0 \longrightarrow \mathrm{Tor}^S_1(M,R) \longrightarrow R^n \xrightarrow{(0 \ *)} R^n \longrightarrow M \longrightarrow 0,$$

so $\mathrm{Tor}^S_1(M,R)$ has a free summand. But this is a contradiction, since this module is isomorphic to M. It follows that all of the $\overline{f\cdot e_j}$ belong to the span of the $\overline{h\cdot e_i}$. This proves the first statement in the lemma and yields the existence of the matrix A.

To prove the equivalence of statements (1)-(4), suppose (1) holds. Then exactly the same proof as above shows that there is an $n\times n$ matrix α with entries in $\mathfrak{m}$ such that $(\alpha \mid g\cdot I_n)$ minimally presents M over T. Thus $\{g\cdot e_j\}$, $1 \le j \le n$, is part of a minimal generating set for Z, so (2) holds. Now suppose (2) holds. Since $\mu(Z) = 2n$

and the $g \cdot e_j$ are part of a minimal generating set for Z, there exists and $n \times n$ matrix β such that $(\beta \mid g \cdot I_n)$ presents M over T. But $\mathrm{rad}(\mathrm{Fitt}_T(M)) = \mathrm{rad}((\det(\beta), g)T)$ has grade two, so $\det(\beta)$ is a non-zerodivisor on S_g, and therefore

$$0 \longrightarrow S_g^n \xrightarrow{\beta} S_g^n \longrightarrow M \longrightarrow 0$$

is exact. Thus, (1) and (2) are equivalent. The proof of the equivalence of (1) and (3) is similar, so (1)-(3) are equivalent. Finally, if g is equivalent to $f - \lambda h$ modulo $\mathfrak{m} I$, then $A - \lambda \cdot I_n$ is the matrix of coefficients expressing the $\overline{g \cdot e_j}$ in terms of the $\overline{h \cdot e_i}$, so the equivalence of (2) and (4) is immediate. Thus (1)-(4) are equivalent and the proof of the lemma is complete. □

Let $g \in I$ be a minimal generator. If g' is any other minimal generator having the same image as g in $I \otimes_T k$, then since $I \cdot T^n \subseteq Z$, the images of the $g \cdot e_j$ in $Z \otimes_T k$ agree with the images of the $g' \cdot e_j$. Thus, Lemma 1.1 shows that M has finite projective dimension over $T/(g)$ if and only if M has finite projective dimension over $T/(g')$ (which we expect, by [A; Theorem 3.9]). Moreover, if the image of g in $I \otimes_T k$ is not a unit multiple of h, then, up to images in $I \otimes_T k$, $g = f - \lambda h$ for an appropriate unit λ. This means that for the case $c = 2$, we have determined the intermediate complete intersections of codimension $c - 1$ over which M has finite projective dimension and that these rings correspond to the elements of k that are not eigenvalues of the transition matrix A.

With the previous lemma in hand, we may give an elementary proof of the following result from [A]. While this result is not stated explicitly, it follows readily from [A; Theorem 3.9] and elementary properties of cones in affine space.

1.2 Corollary. *Suppose that k is algebraically closed. Let R be as above and M be any R-module. Then the following are equivalent :*

(1) $\mathrm{pd}_R(M) < \infty$.
(2) $\mathrm{pd}_S(M) < \infty$*, for all intermediate complete intersections* S.
(3) $\mathrm{pd}_S(M) < \infty$*, for all codimension* $c-1$ *intermediate complete intersections* S.
(4) $\mathrm{pd}_S(M) < \infty$*, for all intermediate hypersurfaces* S.

Proof. We prove that (1)-(4) are equivalent by induction on c. Assume for the moment that the base case $c = 2$ holds and that (1)-(4) are equivalent for all rings R', where R' is an intermediate complete intersection of codimension $d < c$ (with d replacing c in (1)-(4)). Now take $c > 2$. That (1) implies (2) follows easily from reverse induction on $\mathrm{depth}(M)$, starting with the case $\mathrm{depth}(M) = \mathrm{depth}(R)$. That (2) implies (3) is trivial. To see that (3) implies (4), let T' be any intermediate complete intersection of codimension one. Then, we can regard R as a T'-module and as such, it has codimension $c-1$. Any intermediate complete intersection between T' and R of codimension $c-2$ corresponds to an intermediate complete intersection of codimension $c-1$ between T and R, so M has finite projective dimension over such a ring, by hypothesis. Therefore by induction, M has finite projective dimension over T'. Finally, suppose (4) holds. Fix T'', an intermediate complete intersection of codimension $c-1$. Condition (4) and the induction hypothesis imply that M has finite projective dimension over T''. This is true for all such T'', so that if S is any intermediate complete intersection of codimension $c-2$, the $c=2$ case applied to S and R shows that M has finite projective dimension over R.

Now, to prove the base case, suppose $c=2$. Then (1) implies (2) as before. Moreover, conditions (2)-(4) are all the same. So assume (2) holds. Replacing M with a high syzygy over R, we may further assume $\operatorname{depth}(M)=\operatorname{depth}(R)$. Therefore, we want to show that M is free over R. Write $M=G\oplus N$, where G is a free R-module and N has no free summand over R. Suppose $N\neq 0$. Write $I=(f,h)T$ and note that for $S:=T/(h)$, $\operatorname{pd}_S(N)=1$. We now apply the lemma to N. Since k is algebraically closed, we can find $\lambda\in k$ an eigenvalue for the transition matrix A in Lemma 1.1. Thus, for S_λ in Lemma 1.1, $\operatorname{pd}_{S_\lambda}(N)=\infty$, which contradicts the assumption (2). So $N=0$ and M is free, as required. □

1.3 Example. Here is an example showing how the corollary can fail if the residue field is not algebraically closed. Set $T:=\mathbb{Z}_2[[x,y]]$ and $I:=(x^2,y^2)T$. Thus, $R=T/I$ is zero-dimensional, so no R-module has finite projective dimension unless it is a free R-module. To see that a module has finite projective dimension over every intermediate hypersurface between T and R, by the comments following Lemma 1.1, it suffices to check that it has finite projective finite projective dimension over the three intermediate hypersurfaces determined by the polynomials x^2, y^2 and x^2+y^2. Let M be the module presented over R by the matrix $\alpha := \begin{pmatrix} x & y \\ x+y & x \end{pmatrix}$. A straightforward calculation shows that over each of these hypersurfaces, the columns of the matrix $\begin{pmatrix} x^2 & 0 & y^2 & 0 \\ 0 & x^2 & 0 & y^2 \end{pmatrix}$ are in the span of the the columns of α. Thus, α presents M over the given hypersurfaces. On the other hand, $\det(\alpha)=x^2+xy+y^2$, which is a non-zerodivisor in the given rings. Therefore, M has projective dimension one over each of these rings, and therefore over every intermediate complete intersection of codimension one. Moreover, it is interesting to note that if one takes $h:=y^2$ and $f:=x^2$ as in Lemma 1.1, then the transition matrix $A=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. The characteristic polynomial of A is $\lambda^2+\lambda+1$, so A has no eigenvalues over $\mathbb{Z}_2$.

Section 2

In this section we prove one of our main results, Theorem 2.2, which says that M is free over R if and only if $\mathrm{Fitt}_T(M)$ is grade unmixed. This has the surprising consequence that if T satisfies S_{c+1}, then *every* finitely generated non-free module M over R with a rank has the property that $\mathrm{Fitt}_T(M)$ has an embedded component. A test for finite projective dimension over R then follows readily from the theorem. The results in this section (and the next) rely heavily upon the following crucial proposition.

2.1 Proposition. *Let M be a finitely generated R-module,* $\mathrm{rank}(M) = r$. *Suppose that $\underline{h} := h_1, \dots, h_{c-1}$ is part of a minimal generating set for I and set $S := T/(\underline{h})$. Then* $\mathrm{pd}_S(M) = 1$ *if and only if $I_n(\phi) + (\underline{h}) = I^r + (\underline{h})$.*

Proof. We first note that since M_P is free of rank r for all $P \in \mathrm{Ass}(R) = \mathrm{Ass}(T/I)$, $\mathrm{Fitt}_T(M_P) = \mathrm{Fitt}_T(R_P^r) = I_P^r$. Since $\mathrm{Ass}(T/I) = \mathrm{Ass}(T/I^r)$, it follows that $I_n(\phi) \subseteq I^r$, so $I_n(\phi) + (\underline{h}) \subseteq I^r + (\underline{h})$ always holds.

Suppose $\mathrm{pd}_S(M) = 1$. Then there exists an $n \times n$ matrix ψ over T such that

$$0 \longrightarrow S^n \xrightarrow{\psi \otimes 1_S} S^n \longrightarrow M \longrightarrow 0,$$

is exact. Thus, $I_n(\psi)S = I_n(\phi)S$, by invariance of Fitting ideals. Therefore, $(I_n(\phi), \underline{h})T = (\Delta, \underline{h})T$, where $\Delta = \det(\psi)$ is a non-zerodivisor on S. Take P in $\mathrm{Ass}(T/(I_n(\phi), \underline{h}))$. Then $P \in \mathrm{Ass}(T/(\Delta, \underline{h}))$, so $\mathrm{depth}(T_P) = c$. Since $I_n(\phi)$ and I have the same nilradical, P contains I, so $P \in \mathrm{Ass}(T/I)$. Thus, $I_n(\phi)_P = I_P^r$, so $(I^r, \underline{h})_P \subseteq (I_n(\phi), \underline{h})_P$. Since this holds for all $P \in \mathrm{Ass}(T/(I_n(\phi), \underline{h}))$, we have $I^r + (\underline{h}) \subseteq I_n(\phi) + (\underline{h})$, which gives what we want.

Conversely, suppose that $(I_n(\phi), \underline{h})T = (I^r, \underline{h})T$. Then, writing $I = (\underline{h}, l)T$, for some l, $(I^r, \underline{h}) = (l^r, \underline{h})$. Therefore $I_n(\phi)$ is principal modulo $(\underline{h})$. It follows that $(I_n(\phi), \underline{h}) = (\Delta, \underline{h})$ and that $\Delta \equiv \lambda l^r$ modulo $(\underline{h})$, for some $n \times n$ minor Δ of ϕ and unit λ. Let ψ be the $n \times n$ submatrix of ϕ whose determinant is Δ. Over S, let N denote the cokernel of ψ. In other words, we have an exact sequence

$$0 \longrightarrow S^n \xrightarrow{\psi \otimes 1_S} S^n \longrightarrow N \longrightarrow 0.$$

Since the submodule of S^n spanned by the columns of ψ is contained in the one spanned by the columns of ϕ, it follows that N maps surjectively onto M. Let K denote the kernel of this map. We show $K = 0$, which will complete the proof.

Take $P \subseteq T$ such that $P \in \mathrm{Ass}(N)$. It is enough to show that $K_P = 0$. By the Auslander-Buchsbaum formula, $\mathrm{depth}(S_P) = \mathrm{pd}_{S_P}(N_P) + \mathrm{depth}(N_P) = 1$. Since P contains lS, this implies that $P \in \mathrm{Ass}(T/I)$. It follows that M_P is isomorphic to $(S^r/l \cdot S^r)_P$, as S_P-modules. Now, aside from the original presentation matrix ϕ, M is also presented over S_P by the matrix $l \cdot I_r$. By invariance of Fitting ideals, we have $I_{n-i}(\phi)S_P = l^{r-i}S_P$, for $i = 0, \ldots, r$. Since $\operatorname{ann}(N_P) = (\Delta S_P :_{S_P} I_{n-1}(\psi))$ (see [BE]), we have

$$lS_P = (l^r S_P :_{S_P} l^{r-1}) = (\Delta S_P :_{S_P} I_{n-1}(\phi)) \subseteq (\Delta S_P :_{S_P} I_{n-1}(\psi)) = \operatorname{ann}(N_P),$$

so l annihilates N_P. Therefore, we may regard N_P and K_P as R_P-modules. Since M_P is free over R_P, $N_P = M_P \oplus K_P$, as R_P-modules, and also as S_P-modules. Thus, $\operatorname{Fitt}_{S_P}(N_P) = \operatorname{Fitt}_{S_P}(M_P) \cdot \operatorname{Fitt}_{S_P}(K_P)$. Since $\operatorname{Fitt}_{S_P}(N_P) = \operatorname{Fitt}_{S_P}(M_P)$, it follows that $\operatorname{Fitt}_{S_P}(K_P) = S_P$, so $K_P = 0$, which completes the proof of the proposition. □

We may now state and prove the main result of this section.

2.2 Theorem. *Let M be a finitely generated R-module having rank r. Then M is free over R if and only if $\operatorname{Fitt}_T(M)$ is grade unmixed.*

Proof. If M is a free R-module, then $\operatorname{Fitt}_T(M) = \operatorname{Fitt}_T(R^r) = I^r$. Thus, $\operatorname{Fitt}_T(M)$ is a perfect ideal, and perfect ideals are grade unmixed. Conversely, suppose $\operatorname{Fitt}_T(M)$ is grade unmixed. Then any prime P associated to $\operatorname{Fitt}_T(M)$ is a grade c prime containing I and is therefore an associated prime of I. But the proof of Proposition 2.1 shows that $\operatorname{Fitt}_T(M)_P = I_P^r$ for any such prime P, therefore, $\operatorname{Fitt}_T(M) = I^r$. As we would like to invoke Corollary 1.2, we pass to a faithfully flat extension $\tilde{T}$ of T having algebraically closed residue field (see [EGA; Lemma 19.7.1.3]). Note that we still have the equality $\operatorname{Fitt}_{\tilde{T}}(M \otimes_T \tilde{T}) = I^r\tilde{T}$. Setting $\tilde{R} := \tilde{T}/I\tilde{T}$, we have that $\tilde{R}$ is faithfully flat over R. If $M \otimes_R \tilde{R}$ were free over $\tilde{R}$, then M would be free over R and we would be done. Thus, we may assume $T = \tilde{T}$.

Now, let $\underline{h} := h_1, \ldots, h_{c-1}$ be any $c-1$ elements in I forming part of a minimal generating. Since $\operatorname{Fitt}_T(M) = I^r$, $I_n(\phi) + (\underline{h}) = I^r + (\underline{h})$, so by Proposition 2.1, $\operatorname{pd}_{T/(\underline{h})}(M) = 1$. Since this holds for all choices of $\underline{h}$, Corollary 1.2 implies that M has finite projective dimension over R. Moreover, since $\operatorname{depth}_{T/(\underline{h})}(M) = \operatorname{depth}(T/(\underline{h})) - 1$, $\operatorname{depth}(M) = \operatorname{depth}(R)$, so M is free over R. □

The proof of Theorem 2.2 gives a little more than we have stated. We record this as a corollary for future reference.

2.3 Corollary. *Suppose that M has rank r. The following are equivalent :*

(1) *M is a free R-module.*
(2) $\operatorname{Fitt}_T(M)$ *is grade unmixed.*
(3) $\operatorname{Fitt}_T(M)$ *and I have the same associated primes.*
(4) $\operatorname{Fitt}_T(M) = I^r$.

2.4 Corollary. *Suppose T that satisfies Serre's condition S_{c+1} and M has rank r. If M is not free over R, then $\text{Fitt}_T(M)$ has embedded associated primes. In fact, the minimal primes over $I_{n-r}(\phi)$ are embedded primes of $\text{Fitt}_T(M)$.*

Proof. The first statement follows immediately from the theorem. For the second statement, by [E; Prop. 20.7], $I^r \cdot I_{n-r}(\phi) \subseteq I_n(\phi) = \text{Fitt}_T(M)$. On the one hand, since M_P is free of rank r over R for all $P \in \text{Ass}(R)$, $I_{n-r}(\phi)R$ is not contained in any $P \in \text{Ass}(R)$, so $\text{height}(I_{n-r}(\phi)) > c = \text{height}(\text{Fitt}_T(M))$. On the other hand, since $I_n(\phi) \subseteq I^r$, Corollary 2.3 gives that $(I_n(\phi) : I^r)$ defines the locus of primes for which M is not free over R, so $(I_n(\phi) : I^r)$ and $I_{n-r}(\phi)$ have the same radical. Thus, if P is a prime in T minimal over $I_{n-r}(\phi)$, I_P^r is not contained in $I_n(\phi)_P$, so the P_P-primary ideal $I_{n-r}(\phi)_P$ consists of zero divisors modulo $\text{Fitt}_T(M)_P$. Therefore, P is an embedded associated prime of $\text{Fitt}_T(M)$. □

The next corollary is an immediate consequence of Theorem 2.2, and gives a criterion for M to have finite projective dimension over R in terms of a Fitting ideal defined over T.

2.5 Corollary. *Suppose that T satisfies Serre's condition S_{c+1}. Then M has finite projective dimension over R if and only if $\text{Fitt}_T(\Omega_R^{d-\delta}(M))$ has no embedded primes.*

2.6 Remark. In principle, one would like to have a formula for the Fitting ideal of $\Omega_R^{d-\delta}(M)$ in Corollary 2.5, but this seems to require an explicit description of the R syzygies of M, which is tantamount to invoking constructions along the lines of Eisenbud-Shamash (see [A2; section 9]). Unfortunately, these constructions are iterative and do not readily lead to closed form expressions. Another approach, also iterative, is to use a construction from homological algebra that yields a free resolution of the first module in a short exact sequence of modules, given free resolutions of the other two terms. Applying this to the short exact sequence of T-modules $0 \to \Omega_R^1(M) \to R^n \to M \to 0$, the first map in the resulting resolution for $\Omega_R^1(M)$ can then be taken as a presentation from which the Fitting ideal can be calculated. We sketch the construction, for the reader's convenience. Let $(\mathbb{F}, \phi_i)$ be a free resolution of M over T and $(\mathbb{G}, \psi_i)$ be a free resolution of I over T, e.g., the Koszul complex. Then the augmentation of the acyclic complex $\mathbb{G} \otimes F_0$ maps onto M, so the comparison theorem gives a map of complexes $\mathbb{G} \otimes F_0 \xrightarrow{\alpha} \mathbb{F}$. Let $(\mathbb{H}, \rho_i)$ be the mapping cone of α. Then $\mathbb{H}$ is an acylic complex. If we truncate this complex to

$$\cdots \longrightarrow H_3 \xrightarrow{\rho_3} F_2 \oplus (G_1 \otimes F_0) \xrightarrow{(\phi_2|\alpha_1)} F_1,$$

then we still have an acylic complex, and it is easy to see that the map $(\phi_2|\alpha_1)$ gives a presentation for $\Omega_R^1(M)$. Thus, we have a free resolution of $\Omega_R^1(M)$ as a T-module. One can now iterate this construction to produce free resolutions and

Fitting ideals over T of higher R syzygies of M. It should be noted that even if we just want the matrices presenting the $\Omega^i_R(M)$ over T, this construction shows that we need to know the maps further along in the T resolutions of the $\Omega^j_R(M)$, for $j < i$.

We conclude this section with some examples. The first example illustrates the conclusion of Theorem 2.2 (and its corollaries) and the next three examples illustrate the extent to which the hypotheses in Theorem 2.2 are required.

2.7 Example. Let k be a field and $T := k[[x,y,z]]$. Set set $f := xz-y^2$, $g := x^3-z^2$, $I := (f,g)T$ and $R := T/I$. Then R is a one-dimensional complete intersection domain of codimension 2. For $M := k$, by Theorem 2.2 we expect the Fitting ideal of $\Omega^1_R(M) = \mathfrak{m}\, R$ to have an embedded associated prime. The construction in the remark above readily gives the following presentation over T

$$T^5 \xrightarrow{\begin{pmatrix} y & z & 0 & 0 & x^2 \\ -x & 0 & -z & y & 0 \\ 0 & -x & -y & -x & -z \end{pmatrix}} T^3 \to \Omega^1_R(M) \to 0.$$

Thus, $\mathrm{Fitt}_T(\Omega^1_R(M)) = \mathfrak{m}\cdot I$, and $\mathfrak{m}$ is an embedded associated prime.

2.8 Example. Theorem 2.2 fails for $c = 1$. Indeed, let k be a field and x, y, z be indeterminates over k. Set $T := k[[x,y,z]]$, $f := xz - y^2$, $I := fT$ and $R := T/I$. Let M be defined as the cokernel of the map from R^2 to R^2 given by the image of the matrix $\begin{pmatrix} y & x \\ x & y \end{pmatrix}$. Then $\mathrm{depth}(M) = \mathrm{depth}(R)$ (since M is just $\Omega^2_R(k)$) and M has infinite projective dimension over R. On the other hand, it is easy to see that the same matrix presents M over T, so $\mathrm{Fitt}_T(M) = I$ is unmixed.

2.9 Example. Theorem 2.2 fails if I is not generated by a regular sequence. Let k be an infinite field and T be the polynomial ring in $n \cdot m$ variables over k, localized at its homogenous maximal ideal. Let ϕ be the corresponding generic $n \times m$ matrix and assume $m > n$ and $n \geq 2$, so that $I := I_n(\phi)$ is not a complete intersection. Let M denote the cokernel of ϕ over T and take $R := T/I$. Then M is an R-module of maximal depth and has infinite projective dimension. But $\mathrm{Fitt}_T(M) = I_n(\phi)$ is unmixed.

2.10 Example. Theorem 2.2 fails if M does not have a rank. Indeed, just take any case in which R has dimension zero, i.e., T is Cohen-Macaulay and I is generated by a maximal regular sequence. Then for any R-module M, $\mathrm{Fitt}_T(M)$ is $\mathfrak{m}$-primary and therefore unmixed.

Section 3

In this section we demonstrate further connections between the Fitting ideals of modules over R and the property of having finite projective dimension. Let M be an R-module. In our proof of Theorem 2.2, we made crucial use of Corollary 1.2 which guarantees that M has finite projective dimension over R if and only if M has finite projective dimension over every intermediate complete intersection of codimension $c-1$ (if k is algebraically closed). Thus we adopt the view in this section that it is of interest to know whether or not M has finite projective dimension over any intermediate complete intersection of codimension $c-1$, and if so, how are these rings determined. In fact, one answer to this question is given by [A, Theorem 3.9], where finite projective dimension over intermediate complete intersections is determined by the *support variety* of M. We will mention this in a remark below, but for now our goal remains to express this property in terms of various Fitting ideals.

As one can generally replace M by a high syzygy over R, some of the proofs of our results focus on modules M satisfying $\operatorname{depth}(M) = \operatorname{depth}(R)$. Note that in this case, were M to have finite projective dimension over R or any intermediate complete intersection, then M would have finite projective dimension over T and thus M would be a perfect T-module. Before getting to the main result of this section, we begin with a lemma which tells us about the resolution of such an M over T.

3.1 Lemma. *Let M be a perfect T module of grade c. Assume there exists a regular sequence $h_1, \dots, h_{c-1} \in \operatorname{ann}(M)$ such that for $S := T/(h_1, \dots, h_{c-1})T$, M has finite projective dimension over S. Then :*

(1) *There exists an $n \times n$ matrix ψ over T such that the minimal resolution for M over T is $C \otimes_T K$, where C is the complex $0 \longrightarrow T^n \xrightarrow{\psi} T^n \longrightarrow 0$ and K is the Koszul complex on the h_i.*

(2) *Let $g_1, \dots, g_{c-1} \in \operatorname{ann}(M)$ be a regular sequence and suppose there exists an $n \times m$ matrix α over T such that the matrix $(\alpha | g_1 \cdot I_n | \cdots | g_{c-1} \cdot I_n)$ is a minimal presentation of M over T. Then $m = n$ and for $S' := T/(g_1, \dots, g_{c-1})T$, $\operatorname{pd}_{S'}(M) < \infty$.*

Proof. For (1), by grade considerations and the fact that $\operatorname{pd}_S(M) < \infty$, it follows that $\operatorname{pd}_S(M) = 1$. Since $\operatorname{ann}_S(M)$ contains a non-zerodivisor, it follows that there exists an $n \times n$ matrix ψ over T such that

$$0 \longrightarrow S^n \xrightarrow{\psi \otimes_S 1_S} S^n \longrightarrow M \longrightarrow 0,$$

is exact. Therefore, $h_1, \dots, h_{c-1}, \det(\psi)$ form a regular sequence in T. If we set $J := (h_1, \dots, h_{c-1})T$ and $N := \operatorname{coker}(\psi)$, then we have that $\operatorname{Tor}_i^T(T/J, N) = 0$, for all $i > 0$. Since this Tor may also be computed as the homology of $C \otimes_T K$, (1) follows.

For (2), since $C \otimes_T K$ is a minimal resolution of M over T, the $n \times cn$ matrix $(\psi | h_1 \cdot I_n | \cdots | h_{c-1} \cdot I_n)$ is a minimal presentation for M over T. Thus, $\mu(\Omega^1_T(M)) = cn$, so the number of columns in the matrix $(\alpha | g_1 \cdot I_n | \cdots | g_{c-1} \cdot I_n)$ equals cn, so $m = n$. Therefore, $(S')^n \xrightarrow{\alpha \otimes_{S'} 1_{S'}} (S')^n \to M \to 0$ is a minimal presentation of M over S'. On the other hand, the hypothesis on M implies that the grade of the ideal of $n \times n$ minors of $(\alpha | g_1 \cdot I_n | \cdots | g_{c-1} \cdot I_n)$ is c. It follows easily from this that $g_1, \cdots, g_{c-1}, \det(\alpha)$ form a regular sequence. Thus,

$$0 \to (S')^n \xrightarrow{\alpha \otimes_{S'} 1_{S'}} (S')^n \to M \to 0,$$

is exact, so $\mathrm{pd}_{S'}(M) < \infty$. □

The next lemma is a generalization of the first part of Lemma 1.1. We omit the proof as it is analogous to the proof of its counterpart.

3.2 Lemma. *Assume that M is an R-module that does not have a summand isomorphic to R. Set $Z := \Omega^1_T(M) \subseteq T^n$. Assume $h_1, \ldots, h_{c-1}$ are part of a minimal generating set for I and $\mathrm{pd}_S(M) = 1$, for $S := T/(\underline{h})$. Take $f \in I$ such that f together with the h_i generate I. Then the images of $f \cdot e_j$, $1 \le j \le n$, belong to the span of the images of the $h_i \cdot e_j$, $1 \le i \le c-1$, as vectors in $Z \otimes_T k$.*

The next theorem is the main result of this section. It tells us in terms of Fitting ideals when M has finite projective dimension over at least one intermediate complete intersection of codimension $c-1$.

3.3 Theorem. *Suppose that k is infinite and M does not have finite projective dimension over R. Let $\Omega := \Omega^{d-\delta+1}_R(M)$ denote the $d - \delta + 1$ syzygy in a minimal free resolution of M over R and assume* $\mathrm{rank}(\Omega) = r$. *Write W for the subspace of $I^r \otimes_T k$ spanned by the image of* $\mathrm{Fitt}_T(\Omega)$. *Then the following are equivalent :*

(1) *M has finite projective dimension over some intermediate complete intersection of codimension $c-1$.*
(2) $\dim_k(W) \ge 1$.
(3) $\dim_k(W) = 1$.

Proof. Without loss of generality, we may replace M by Ω and assume $\mathrm{rank}(M) = r$, $\mathrm{depth}(M) = \mathrm{depth}(R)$ and that M has no summand isomorphic to R (see [A2; Corollary 1.2.5]). Now suppose $h_1, \ldots, h_{c-1} \in I$ are part of a minimal generating set for I and $\mathrm{pd}_S(M) < \infty$, for $S := T/(h_1, \ldots, h_{c-1})T$. Then $\mathrm{pd}_S(M) = 1$, so by Proposition 2.1, $I_n(\phi) + (\underline{h}) = I^r + (\underline{h})$, which implies $I_n(\phi) \not\subseteq \mathfrak{m} I^r$. Indeed, suppose $I_n(\phi) \subseteq \mathfrak{m} I^r$. Then $I^r \subseteq (\underline{h},\ mI^r)T$. Choose $f \in I$ such that $I = (f, h_1, \ldots, h_{c-1})T$. Then I^r is generated by $f^{r-1}(\underline{h}), f^{r-2}(\underline{h})^2, \ldots, (\underline{h})^r$, which is a contradiction. Thus $I_n(\phi) \not\subseteq \mathfrak{m} I^r$, so $\dim_k(W) \ge 1$.

Conversely, suppose $\dim_k(W) \geq 1$. Let $p \in I_n(\phi) \setminus \mathfrak{m} I^r$ and write p' for the image of p in W. Now, we can think of $I^r \otimes_T k$ as the degree r piece of the fiber ring F of I, which is a polynomial ring in c variables over k. Thus p' is a homogenous form of degree r. Since k is infinite, we may perform a change of variables in F to assume that there is a linear form l' in F such that $p' = (l')^r + q'$, for q' a form of degree r not having $(l')^r$ as one of its monomials. Let $h'_1, \ldots, h'_{c-1}$ be linear forms in F such that $F = k[h'_1, \ldots, h'_{c-1}, l']$ and let $h_1, \ldots, h_{c-1}$ denote their pre-images in I. Then $h_1, \ldots, h_{c-1}$ form part of a minimal generating set for I, and by construction, $(p) + (\underline{h}) = I^r + (\underline{h})$. Thus $I_n(\phi) + (\underline{h}) = I^r + (\underline{h})$, so M has finite projective dimension over the intermediate complete intersection $S := T/(\underline{h})$, by Proposition 2.1. Thus (1) and (2) are equivalent. It remains to show that (2) implies (3).

Suppose $\dim_k(W) \geq 1$. Then there is an $n \times n$ submatrix ψ of ϕ such that $\det \psi$ has a non-zero image in W. We now make the following claim :

Claim : For each $1 \leq i \leq r-1$, $I_{n-i}(\psi) \subseteq \mathfrak{m} I^{r-i}$.

Before proving the claim, we note that for each i invariance of higher order Fitting ideals gives $I_{n-i}(\psi) \subseteq I^{r-i}$ (since M_P is free of rank r for all $P \in \mathrm{Ass}(R)$) . We now prove the claim by induction on i. Suppose $i = 1$ and $I_{n-1}(\psi) \nsubseteq \mathfrak{m} I^{r-1}$. Without loss of generality, we may assume that the determinant of the $(n-1) \times (n-1)$ submatrix ψ' of ψ consisting of the first $n-1$ rows and columns does not belong to mI^{r-1}. Then the images p and q of $\det(\psi)$ and $\det(\psi')$ in the fiber ring F are forms of degree r and $r-1$. Because k is an infinite field, we may find linear forms $h'_1, \ldots, h'_{c-1}$ in F such that p and q have non-zero images in $F/(\underline{h'})$. Let $f' \in F$ be a linear form in F such that $F = k[f', h'_1, \ldots, h'_{c-1}]$. Then as in the previous paragraph, M has finite projective dimension over the the intermediate complete intersection $S := T/(\underline{h})$. In fact, the minimal resolution for M over S is given by

$$0 \longrightarrow S^n \xrightarrow{\psi \otimes_S 1_S} S^n \longrightarrow M \longrightarrow 0,$$

where as elements of S, $\det(\psi \otimes_S 1_S) = uf^r$ and $\det(\psi' \otimes_S 1_S) = vf^{r-1}$, for units u, v. Let $s_1, \ldots, s_n \in S$ denote the elements along the n^{th} column of $\psi \otimes_S 1_S$. Then $uf^r = \det(\psi \otimes_S 1_S) = s_1\delta_1 + \cdots + s_{n-1}\delta_{n-1} + s_n \det(\psi' \otimes_S 1_S)$, for suitable minors δ_i of $\psi \otimes_S 1_S$. Since each δ_i is divisible by f^{r-1} in S (since $I_{n-1}(\phi) \subseteq I^{r-1}$), we obtain $uf = s_1t_1 + \cdots + s_{n-1}t_{n-1} + s_n v$. It follows that after elementary row operations we may assume that $\psi \otimes_S 1_S$ still has $\psi' \otimes_S 1_S$ as the upper left $(n-1) \times (n-1)$ block and f as its (n, n)-entry. Thus we obtain a presentation (necessarily minimal) over T for M of the form

$$[* \; f \cdot e_n | h_1 \cdot I_n | \cdots | h_{c-1} \cdot I_n].$$

It follows that the image of $f \cdot e_n$ is not in the span of the images of the $h_i \cdot e_j$ in

$Z \otimes_T k$, for $Z := \Omega^1_T(M)$. But this contradicts Lemma 3.2. Therefore, we must have $I_{n-1}(\psi) \subseteq \mathfrak{m} I^{r-1}$ and the case $i = 1$ of the claim has been shown.

Now suppose that $i > 1$ and $I_{n-i}(\psi) \not\subseteq \mathfrak{m} I^{r-i}$. Then there exists an $(n-i) \times (n-i)$ minor of ψ whose image w' in F corresponds to a non-zero form of degree $r - i$. As before, we select linear forms $h'_1, \dots, h'_{c-1} \in F$ such that p and w' are non-zero in $F/(\underline{h'})$. Then, for $S := T/(\underline{h})$, M has finite projective dimension over S and its resolution is given by $\psi \otimes_S 1_S$. For f such that $I = (f, \underline{h})$,

$$f^{i-1} \cdot I_{n-i}(\psi \otimes_S 1_S) \subseteq I_{n-1}(\psi \otimes_S 1_S)$$

over S, by ([E; Prop. 20.7]). Thus,

$$f^{i-1} \cdot I_{n-i}(\psi) \subseteq I_{n-1}(\psi) + (\underline{h}) \subseteq \mathfrak{m} I^{r-1} + (\underline{h}).$$

It follows immediately from this that $f^{i-1} \cdot I_{n-i}(\psi) \subseteq \mathfrak{m} I^{r-1} + (\underline{h}) I^{r-2}$. This implies that $(f^{i-1})' \cdot w'$ is zero in $F/(\underline{h'})$, which is a contradiction. Thus, $I_{n-i}(\psi) \subseteq \mathfrak{m} I^{r-i}$, for all i in the required range and the Claim has been verified.

To complete the proof that $\dim_k(W) = 1$, we take any $h_1, \dots, h_{c-1}$ forming part of a minimal generating set for I such that for $S := T/(\underline{h})$, M has finite projective dimension over S and its minimal resolution is given by $\psi \otimes_S 1_S$. Then by Lemma 3.1, we may assume that $\phi = (\psi | h_1 \cdot I_n | \cdots | h_{c-1} \cdot I_n)$. Thus, $I_n(\phi)$ is generated by $\det(\psi)$ together with the ideals $\underline{h}^i \cdot I_{n-i}(\psi)$, for $i = 1, \dots, n$. By the Claim, $I_{n-i}(\psi) \subseteq \mathfrak{m} I^{r-i}$ for $1 \leq i \leq r-1$. Therefore $\underline{h}^i \cdot I_{n-i}(\psi) \subseteq \mathfrak{m} I^r$, for $1 \leq i \leq r-1$. For $r \leq i \leq n$, we clearly have that $\underline{h}^i \cdot I_{n-i}(\psi) \subseteq \mathfrak{m} I^r$. It now follows that the image of $\det(\psi)$ in W spans W, so $\dim_k(W) = 1$ as desired. This completes the proof of the theorem. □

3.4 Remark. In Theorem 3.3 we cannot replace $\Omega_R^{d-\delta+1}(M)$ by $\Omega_R^{d-\delta}(M)$ if we wish to include the stronger statement that $\dim_k(W) = 1$. This is because $\Omega_R^{d-\delta+1}(M)$ does not have a free summand, while $\Omega_R^{d-\delta}(M)$ could. Indeed, suppose $\delta = d$ and we could write $M = N \oplus R$ as R-modules. If it were the case that the dimension of the image of $\mathrm{Fitt}_T(N)$ in $I^{r-1} \otimes_T k$ were one, then since $\mathrm{Fitt}_T(M) = \mathrm{Fitt}_T(N) \cdot I$, it would follow that the dimension of the image of $\mathrm{Fitt}_T(M)$ in $I^r \otimes_T k$ would be c. However, the proof of the theorem shows that taking $\Omega_R^{d-\delta}(M)$ suffices for the equivalence of (a) and (b).

The results we have obtained about finite projective dimension in codimension $c - 1$ take particularly nice forms when we assume the $\mathrm{depth}(M) = \mathrm{depth}(R)$. For in this case we just have to consider the Fitting ideal of M itself. The next two corollaries summarize what we have obtained for such a module. The first of these corollaries tells when M has finite projective dimension over some intermediate complete intersection of codimension $c-1$ and relates it to another well-known condition, while the second tells how to obtain all other codimension $c-1$ intermediate complete intersections over which M has finite projective dimension, once we know of one such ring.

3.5 Corollary. *Let k be infinite and M be an R-module having rank r, infinite projective dimension and no free summand. If* $\operatorname{depth}(M) = \operatorname{depth}(R)$*, the following are equivalent :*

(1) *M has finite projective dimension over some intermediate complete intersection of codimension $c-1$.*
(2) *M has a periodic resolution over R.*
(3) $\operatorname{Fitt}_T(M) \not\subseteq \mathfrak{m}\, I^r$.

Proof. The equivalence of (1) and (2) is well-known, and follows from the results in [A] (c.f. Corollaries 3.12 and 4.5). That (1) and (3) are equivalent follows from Theorem 3.3 and the remark above. □

For M in Corollary 3.5, if $\operatorname{Fitt}_T(M) \not\subseteq \mathfrak{m}\, I^r$, the proof of Theorem 3.3 shows how to obtain a ring in codimension $c-1$ over which M has finite projective dimension. Let Δ be a maximal minor of ϕ not in mI^r. Then any minimal generating set $f, h_1, \dots, h_{c-1}$ having the property that $\Delta = \lambda f^r + g$, where $g \in I^r \cap (\underline{h}) + \mathfrak{m}\, I^r$, for a unit λ, is such that M has finite projective dimension over $T/(\underline{h})$. If $f_1, \dots, f_c$ is the original set of generators, then such a generating set can be gotten by taking the generating set $f_1, f_2 - \epsilon_2 f_1, \dots, f_c - \epsilon_c f_1$, for appropriate units ϵ_i.

We will use the following notation in the next corollary. Let $f, h_1, \dots, h_{c-1}$ be a minimal generating set for I and assume $\operatorname{pd}_S(M) = 1$, for $S := T/(\underline{h})$. Let $g_1, \dots, g_{c-1}$ also be part of a minimal generating set for I and write S_g for $T/(\underline{g})$. For each $1 \leq i \leq c-1$, we can write $g_i = \alpha_i \cdot f + \beta_{i,1} \cdot h_1 + \cdot \beta_{i,c-1} \cdot h_{c-1} + p$, with the α_i, $\beta_{r,s}$ units and $p \in \mathfrak{m}\, I$. By Lemma 3.2, the images of the $f \cdot e_j$ belong to to the span of the images of the $h_i \cdot e_j$ in $Z \otimes_T k$, for $Z := \Omega^1_T(M) \subseteq T^n$. Let $A_1, \dots, A_{c-1}$ be $n \times n$ matrices over k such that in $Z \otimes_T k$ the equation $f \cdot I_n = h_1 \cdot A_1 + \cdots + h_{c-1} \cdot A_{c-1}$ holds. Let $A(\alpha, \beta)$ denote the $(c-1)n \times (c-1)n$ matrix consisting of $(c-1) \times (c-1)$ blocks arranged so that the $(i,j)^{\text{th}}$ block is the $n \times n$ matrix $\alpha_i \cdot A_j + \beta_{ij} \cdot I_n$ Then $A(\alpha, \beta)$ is the matrix of coefficients obtained by expressing the images in $Z \otimes_T k$ of the vectors $g_i \cdot e_j$ in terms of the images of the vectors $h_i \cdot e_j$.

3.6 Corollary. *In addition to the notation and assumptions of the preceding paragraph, suppose that M is an R-module of rank r having no free summand. Suppose further that* $\operatorname{depth}(M) = \operatorname{depth}(R)$*. Fix a maximal minor Δ of ϕ not in $\mathfrak{m}\, I^r$. Then the following are equivalent :*

(1) $\operatorname{pd}_{S_g}(M) < \infty$.
(2) *The image of the set $\{g_i \cdot e_j\}_{i,j}$ in $Z \otimes k$ is linearly independent.*

(3) $\mu(\Omega^1_{S_g}(M)) = n$.
(4) $A(\alpha, \beta)$ *has maximal rank.*
(5) $(\Delta) + (\underline{g}) = I^r + (\underline{g})$.
(6) *The image of* Δ *in the fiber ring of* I *is not in the ideal generated by the images of the* g_i.

Proof. Using Lemma 3.1, the proof of the equivalence of (1)-(4) is essentially the same as the proof of the second statement in Lemma 1.1. Now assume (1) holds, so that M has projective dimension one over S_g. As in the proof of Proposition 2.1, there is an $n \times n$ matrix ψ over T such that $\psi \otimes 1_{S_g}$ gives the minimal resolution of M over S_g and $I^r + (\underline{g}) = (\det(\psi)) + (\underline{g})$. By Theorem 3.3, $\det(\psi) = \lambda\Delta + t$, where λ is a unit and $t \in \mathfrak{m} I^r$. Thus, $I^r + (\underline{g}) = (\lambda\Delta + t, \underline{g}) \subseteq (\Delta, \underline{g}) + \mathfrak{m} I^r$, so (5) holds, by Nakayama's lemma. That (5) implies (1) follows from Proposition 2.1. The equivalence of (1) and (6) follows from the proof of Theorem 3.3. □

Note that it follows from Corollary 3.6 that the intermediate complete intersections of codimension $c-1$ over which M has finite projective dimension are precisely those whose ideal generators give rise to a transition matrix $A(\alpha, \beta)$ over k having maximal rank.

When k is algebraically closed, we have the following complementary statement to Theorem 3.3. In 3.7, we retain the notation and hypotheses from 3.3.

3.7 Proposition. *Assume* k *is algebraically closed. There exist* $h_1, \dots, h_{c-1}$ *in* I, *part of a minimal generating set, so that for* $S := T/(\underline{h})$, M *has infinite projective dimension over* S *if and only if* $\dim(W) < \dim_k(I^r \otimes_T k)$.

Proof. Indeed, by Corollary 1.2, M has infinite projective dimension over some such S if and only if M has infinite projective dimension over R if and only if $\Omega_R^{d-\delta+1}(M)$ has infinite projective dimension over R if and only if $\mathrm{Fitt}_T(\Omega_R^{d-\delta+1}(M)) \subsetneq I^r$ (by Corollary 2.3) if and only if $\dim_k(W) < \dim_k(I^r \otimes_T k)$. □

Several of our results can be interpreted in terms of the support variety of M. To do this, we recall the definition and some facts established in [A]. For the sake of convenience, we will also assume that k is algebraically closed. Let V denote the affine space over k determined by the vector space $I \otimes_T k$. If one fixes a minimal generating set $f_1, \dots, f_c$ for I, one obtains a basis for V. Any $h \in I \setminus \mathfrak{m} I$ determines a point (or line through the origin) in V and similarly, any subset $h_1, \dots, h_t$ of a minimal generating set determines a linear subvariety. Without going into detail, the support variety X of M is the algebraic subset of V defined by the homogenous ideal in the polynomial ring of Eisenbud operators that annihilates the graded module $\mathrm{Ext}^*_R(M, k)$. More concretely, it follows from (3.9)-(3.11) in [A] that X consists of the points in V that correspond to the intermediate hypersurfaces over which M has infinite projective dimension.

3.8 Corollary. *Suppose k is algebraically closed. Taking syzygies from a minimal free resolution of M over R, assume that $s := \mathrm{rank}(\Omega_R^{d-\delta}(M))$ and $r := \mathrm{rank}(\Omega_R^{d-\delta+1}(M))$. Let U and W respectively denote the images of the Fitting ideals over T of these modules in the vector spaces $I^s \otimes_T k$ and $I^r \otimes_T k$. Then the following statements hold for the support variety X of M.*

(1) $\dim(X) = 0$ *if and only if* $\dim_k(U) = \dim_k(I^s \otimes_T k)$.
(2) $\dim(X) = 1$ *if and only if* $\dim_k(W) = 1$.
(3) $\dim(X) > 1$ *if and only if* $\dim_k(U) < \dim_k(I^s \otimes_T k)$ *and* $\dim_k(W) = 0$.

Proof. From the preceding paragraph, $\dim(X)$ equals zero if and only if M has finite projective dimension over R, and from Corollary 2.3, this holds if and only if $\mathrm{Fitt}_T(\Omega_R^{d-\delta}(M)) = I^s$. For (2), by the geometry of cones in affine space, $\dim(X) = 1$ if and only if there exists a $(c-1)$-dimensional linear subvariety L that intersects X only at the origin. The latter happens if and only if there is a $(c-1)$-dimensional linear variety L so that every line in L intersects X at the origin, which in turn happens if and only if M has finite projective dimension over each of the corresponding hypersurfaces (by the comments above). By Corollary 1.2, this happens if and only if M has finite projective dimension over the intermediate complete intersection of codimension $c-1$ corresponding to L, and therefore (2) holds by Theorem 3.3. The equivalence in (3) is an immediate consequence of (1), (2) and Theorem 3.3. □

References

[A] L. L. Avramov, *Modules of finite virtual projective dimension*, Invent. Math. **96** (1989), 71-101.

[A2] L.L. Avramov, *Infinite free resolutions*, Six Lectures on Commutative Algebra (1998), Birhkaüser, 1-63.

[BE] D. Buchsbaum and D. Eisenbud, *What annihilates a module?*, J. Algebra **47** (1977), 231-243.

[E] D. Eisenbud, *Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Mathematics, Springer Verlag **150** (1995).

[EGA] A. Grothendieck, *Eléments de géométrie algébrique*, Publ. Math. I.H.E.S. **IV** (1964).

Department of Mathematics, University of Kansas, Lawrence, KS, 66045
E-mail address: huneke@math.ukans.edu

Department of Mathematics, University of Texas at Arlington, Arlington TX, 76019
E-mail address: djorgens@math.uta.edu

Department of Mathematics University of Kansas, Lawrence, KS, 66045
E-mail address: dlk@math.ukans.edu