Let \( I_1, \ldots, I_g \) be regular ideals in a Noetherian ring \( R \). Then it is shown that there exist positive integers \( k_1, \ldots, k_g \) such that 
\[
(I_1^{m_1} \cdots I_g^{m_g}) : (I_1^{n_1} \cdots I_g^{n_g}) = I_1^{n_1} \cdots I_g^{n_g}
\]
for all \( n_i \geq k_i \) \((i = 1, \ldots, g)\) and for all nonnegative integers \( m_1, \ldots, m_g \). Using this, it is shown that if \( \Delta \) is a multiplicatively closed set of nonzero ideals of \( R \) that satisfies certain hypotheses, then the sets \( \text{Ass}(R/(I_1^{n_1} \cdots I_g^{n_g})) \) are equal for all large positive integers \( n_1, \ldots, n_g \). Also, if \( R \) is locally analytically unramified, then some related results for general sets \( \Delta \) are proved.

Introduction

Let \( R \) be a Noetherian ring. It is known that if \( J \) is an ideal of \( R \), then the two sequences of sets \( \text{Ass}(R/J), \text{Ass}(R/J^2), \ldots \) and \( \text{Ass}(R/J^3), \text{Ass}(R/(J^2)^3), \ldots \) eventually stabilize to sets denoted \( \overline{A}^*(I) \) and \( \overline{A}^*(I) \) respectively (see [2, Corollary 1.5 and Proposition 3.4]). Here \( J_d \) denotes the integral closure of \( J \). In Section 1 these results are extended in two directions. It is shown that if \( I_1, \ldots, I_g \) are (regular) ideals of \( R \) and \( \Delta \) is a multiplicatively closed set of ideals satisfying certain hypotheses, then asymptotic stability holds for the sets \( \text{Ass}(R/(I_1^{n_1} \cdots I_g^{n_g})) \), where \( n_1, \ldots, n_g \in \mathbb{N} \) and \( J_\Delta \) is the \( \Delta \)-closure of an ideal \( J \) (see below). For appropriate choices of \( \Delta \) one concludes that the sets \( \text{Ass}(R/I_1^{n_1} \cdots I_g^{n_g}) \) and \( \text{Ass}(R/(I_1^{n_1} \cdots I_g^{n_g}))_\Delta \) enjoy asymptotic stability. In Section 2 we consider the situation for general \( \Delta \)-closures under the hypothesis that \( R \) is locally analytically unramified.

* The first author was partially supported by the General Research Fund at the University of Kansas.

** The third author was supported in part by the National Science Foundation, grant DMS-8521058.
1. Asymptotic stability of \( \text{Ass} \, R/I_1^n \ldots I_g^n \)

We begin by fixing some notation.

**Notation.** Throughout \( R \) will be a Noetherian ring, \( g \) a fixed positive integer and \( I_1, \ldots, I_g \) ideals of \( R \). \( \mathbb{N}_g \) will be the set of all \( g \)-tuples of non-negative integers. If \( \mathbf{n} = (n_1, \ldots, n_g) \in \mathbb{N}_g \), then by \( I_\mathbf{n} \) we mean \( I_1^{n_1} \ldots I_g^{n_g} \). For \( 1 \leq i \leq g \), \( n(i) \) will refer to the \( i \)th component of \( \mathbf{n} \). Also, we will write \( \mathbf{n} \geq \mathbf{m} \) (respectively \( \mathbf{n} > \mathbf{m} \)) if \( n(i) \geq m(i) \) (respectively, \( n(i) > m(i) \)) for all \( 1 \leq i \leq g \). If \( \mathbf{n} \) and \( \mathbf{m} \) are in \( \mathbb{N}_g \) and \( h \geq 0 \) is an integer, then \( h \mathbf{n} \) and \( \mathbf{n} \pm \mathbf{m} \) will be defined in the usual component-wise manner (\( \mathbf{n} - \mathbf{m} \) only being defined when \( \mathbf{n} \geq \mathbf{m} \)). We shall denote by \( J, \) the integral closure of an ideal \( J \) and by \( J^* \) the eventual stable value of \( (J^k : J^k) \subseteq \cdots \). \( J^* \) was introduced in \([5]\), and in \([2, \text{Lemma 8.2}]\) it is shown that if \( J \) is a regular ideal, then \( (J^n)^* = J^n \) for \( n \) large. Both of these operations are special cases of a more general operation, the so-called \( \Delta \)-closure operation, introduced by the third author in \([4]\).

**Definition.** Let \( J \) be an ideal in \( R \) and \( \Delta \) a multiplicatively closed set of non-zero ideals of \( R \). The ascending chain condition guarantees that the set \( \{(JK : K) \mid K \in \Delta \} \) has maximal elements, and since for \( K \) and \( L \) in \( \Delta \), \( (JKL : KL) \) contains both \( (JK : K) \) and \( (JL : L) \), we see that the set under consideration in fact contains a unique maximal element. Let \( J, \Delta \) denote that unique maximal element. The following lemma shows that the notion of \( \Delta \)-closure allows one to discuss simultaneously the asymptotic behavior of \( \text{Ass} \, R/J^n \) and \( \text{Ass} \, R/(J^n)^* \).

1.1. **Lemma.** Let \( \Delta \) be a multiplicatively closed set of non-zero ideals.

(a) If every ideal in \( \Delta \) is regular, then for any ideal \( J \), \( J, \Delta \subseteq J, \Delta \).

(b) If \( \Delta \) equals the set of all regular ideals and \( J \) is regular, \( J, \Delta = J, \Delta \).

(c) If \( J \) is a regular ideal and \( \Delta = \{J^n \mid n \in \mathbb{N}\} \), then \( (J^n)^* = (J^n)^* \) for all \( n \) and \( (J^n), \Delta = (J^n)^* \) for all large \( n \).

**Proof.** The proofs are easy, but we include them for the convenience of the reader. For (a), \( J, \Delta = (JK : K) \) for some \( K \in \Delta \). Suppose \( K \) is generated by \( k_1, \ldots, k_n \). Then for \( x \in J, \Delta \) and \( 1 \leq i \leq n \) we have \( x \cdot k_i = \sum_{j=1}^n a_{ij} k_j \) for \( a_{ij} \in J \). Now a standard determinant argument shows \( x \in J, \Delta \). For (b), suppose \( \Delta \) is the set of all regular ideals and \( J \) is regular. Thus \( x(J, \Delta)^n \subseteq J(J, \Delta)^n \), so \( xK(J, \Delta)^n \subseteq JK(J, \Delta)^n \). Since \( (J, \Delta) \subseteq \Delta \), it follows that \( J, \Delta = (JK(J, \Delta)^n : K(J, \Delta)^n) \), so \( x \in J, \Delta \). Thus \( J, \Delta \subseteq J, \Delta \) and equality holds by part (a). For (c), let \( J \) be a regular ideal and \( \Delta = \{J^n \mid n \in \mathbb{N}\} \). Then \( (J^n)^* = (J^n)^* \) for large \( h \). Thus \( (J^n)^* = (J^n(J^n)^h : J^n)^h \subseteq (J^n)^* \). On the other hand, \( (J^n)^* = (J^n)^* \) for some \( k \), so \( (J^n), \Delta = (J^n)^* \) for some \( k \), \( (J^n)^* = (J^n)^* \) for all large \( n \). Thus \( (J^n), \Delta = (J^n)^* \) and the second part of (c) follows from \([2, \text{Lemma 8.2}]\). \( \square \)
Ideals of the form \((J^{n+1}:J)\) play a vital role in discussing the behavior of various prime divisors associated to large powers of \(J\). The following lemma and proposition will play analogous roles in determining the corresponding behavior of the prime divisors associated to the product of large powers of \(I_1, \ldots, I_g\). In fact, we consider part (c) of Proposition 1.4 to be one of the main results of this paper.

1.2. Lemma. Let \(I_1, \ldots, I_g\) be regular ideals.

(a) Suppose \(n\) and \(m\) are in \(\mathbb{N}_g\) with \(n \geq (1, \ldots, 1)\). Let \(k\) be an integer with \(kn \geq m\). Then \((I^{n+m}:I^m) \subseteq ((I^n)^{k+1}: (I^n)^k) \subseteq (I^n)^*\).

(b) If we set \(\Delta = \{m \mid m \in \mathbb{N}_g\}\), then for \(n \geq (1, \ldots, 1)\), \((I^n)^* = (I^n)_\Delta\).

Proof. For (a), suppose \(x \in (I^{n+m}:I^m)\). Since \(kn-m \in \mathbb{N}_g\), we may write \((I^n)^k = I^{m(kn-m)}\). Thus \(x(I^n)^k = xI^{m(kn-m)} \subseteq I^{n+m(kn-m)} = (I^n)^{k+1}\). This gives the first containment of the conclusion. The second containment is by the definition of \((I^n)^*\).

For (b), suppose \(\Delta = \{m \mid m \in \mathbb{N}_g\}\) and \(n \geq (1, \ldots, 1)\). Then for large integers \(h\), \((I^n)^* = ((I^n)^{h+1}: (I^n)^h) = (I^h I^{hn}: I^{hn}) \subseteq (I^n)_\Delta\), by the definition of \((I^n)_\Delta\). For the reverse inclusion, there is an \(m \in \mathbb{N}_g\) with \((I^n)_\Delta = (I^{n+m}:I^m)\). By the first part of the lemma, this last ideal is contained in \((I^n)^*\).

1.3. Remark. (a) Note that \(k = \max\{m(i) \mid 1 \leq i \leq g\}\) satisfies the hypothesis of Lemma 1.2(a).

(b) In Lemma 1.2, if we do not have \(n \geq (1, \ldots, 1)\), we cannot be assured that \((I^{n+m}:I^m) \subseteq (I^n)^*\). By [5, (3.4) and (4.2)], there exist regular ideals \(I_1, I_2\) such that \((I_1 I_2 : I_1 I_2) = (I_1^*) \subseteq (I^n)^*\). Let \(n = (1,0)\) and \(m = (0,1)\). Then \((I^n)^* = (I_1 I_2 : I_2) \not\subseteq (I_1^*) = (I^n)^*\).

1.4. Proposition. Let \(I_1, \ldots, I_g\) be ideals of \(R\). Fix \(1 \leq i \leq g\). For each \(s \in \mathbb{N}_g\) write \(J^s = I_1^{s_1} \cdots I_i^{s_i} I_{i+1}^{s_{i+1}} \cdots I_g^{s_g}\).

(a) For a finitely generated \(R\) module \(M\) and submodule \(N \subseteq M\), there exists \(k_j \in \mathbb{N}\) such that for all \(n_j \geq k_j\), \(I_1^{n_j} J^s M \cap N = I_1^{n_j-k_j} (I_1^{k_j} J^s M \cap N)\) for all \(s \in \mathbb{N}_{g-1}\).

(b) There exists \(l_j \in \mathbb{N}\) such that \((I_1^{n_j} J^s : I_1^h) \cap I_1^{l_j} J^s = I_1^{l_j} J^s\) for all \(n_j > l_j\), \(h \in \mathbb{N}\) and \(s \in \mathbb{N}_{g-1}\).

(c) If \(I_1\) is a regular ideal, there exists \(d_j \in \mathbb{N}\) such that \((I_1^{n_j} J^s : I_1^h) = I_1^{d_j} J^s\) for all \(n_j > d_j\), \(h \in \mathbb{N}\) and \(s \in \mathbb{N}_{g-1}\). Consequently, there exists \(k \in \mathbb{N}_g\) such that \((I^{n+m}:I^m) = I^n\) for all \(n > k\) and \(m \in \mathbb{N}_g\) (if each \(I_i\) is regular).

Proof. Let \(t_1, \ldots, t_g\) be indeterminates and set \(\mathcal{R} = R[I_1 t_1, \ldots, I_g t_g]\), the Rees ring of \(R\) with respect to \(I_1, \ldots, I_g\). Let \(\mathcal{M} = \mathcal{R} \otimes M\) and \(\mathcal{N}\) be the submodule consisting of all finite sums of the form \(\sum a_r t'_r\) where \(a_r \in I^r M \cap N\) (here we are writing \(t'_r\) for \(t_i^{s_i} \cdots t_g^{s_g}\) if \(r \in \mathbb{N}_g\)). Then \(\mathcal{M}\) is an \(\mathbb{N}_g\)-graded finitely generated \(\mathcal{R}\)-module and \(\mathcal{N}\) has a system of homogeneous generators. As in the proof of the usual Artin-Rees Lemma, let \(k_i\) be the maximum value achieved by any exponent of \(t_i\) in any one of
the generators. Then it is readily seen that the conclusion of (a) holds for this $k_i$.

For (b) let $\mathfrak{B} = (I_i : R : t_i)$ in $\mathfrak{R}$. A brief computation shows that $\mathfrak{B}$ is an $\mathbb{N}_g$-homogeneous $\mathfrak{R}$-ideal, so it has a generating set of the form $a_i t_i^{r_i}, \ldots, a_i t_i^{r_s}$, where $t_i \in \mathbb{N}_g$ and $a_i \in I_i$. Let $I_i = \{\max r_j(i) | 1 \leq j \leq s\} + 1$ and suppose $ct' \in \mathfrak{B}$ satisfies $r(i) > l_i$.

We may write $ct' = \sum_j (b_j t_i^{-r_j}) (a_j t_i^{r_j})$ for elements $b_j t_i^{-r_j} \in \mathfrak{B}$. The choice of $r$ forces each $b_j t_i^{-r_j} \in (I_i : t_i) \mathfrak{R}$ so $ct' \in I_i \mathfrak{R}$.

Now suppose $n_i \in \mathbb{N}$ satisfies $n_i > l_i$. Let $s \in \mathbb{N}_{g-1}$ and suppose $ct' \subseteq I_i^{n_i + 1} J^s$, for $c \in I_i J^s$. Then, writing $t^s$ for $t_1^{r_1} \cdots t_g^{r_g}$, we have $(ct' t^s)(I_i t_i) \subseteq I_i^{n_i + 1} J^s t_i^{n_i + 1} t^s \subseteq I_i \mathfrak{R}$ (since $n_i > l_i$). By the preceding paragraph, $ct' t^s \in I_i \mathfrak{R}$ so $c \in I_i^{l_i + 1} J^s$. We may now repeat the argument until $c \in I_i^n J^1$ as desired. This shows $\mathfrak{B}$ is $\mathfrak{R}$-homogeneous, and the rest of (b) follows from this. To finish, let $a_1, \ldots, a_s$ be a set of regular elements generating $I_i$. As in the proof of [3, Proposition 11(e)] set $M = R \cdot (1/a_1) \oplus \cdots \oplus R \cdot (1/a_s)$ (considered as a submodule of $K \oplus \cdots \oplus K$, for $K$ the total quotient ring of $R$) and $N = \{(r/1, \ldots, r/1) | r \in R\}$. From part (a) there is $k_i \in \mathbb{N}$ such that $I_i J^k M \cap N = I_i^{n_i - k_i} (I_i) J^k M \cap N$ for all $n_i \geq k_i$, and $s \in \mathbb{N}_{g-1}$. It follows readily that $(I_i^n J^s : I_i) = I_i^{n_i - k_i} (I_i J^k : I_i) \subseteq I_i^{k_i} J^s$, for $n_i > k_i$. Since we may increase $k_i$ so that it is larger than $l_i$, for $l_i$ as in part (b), it follows that $(I_i^{n_i + h} J^s : I_i) = I_i^{n_i + h} J^s$ for all large $n_i$, $h \in \mathbb{N}$ and $s \in \mathbb{N}_{g-1}$. The second statement follows from this. \hfill \Box

1.5. Corollary. Let $I_1, \ldots, I_g$ be regular ideals. There is a $d \in \mathbb{N}_g$ such that for all $n \in \mathbb{N}_g$ with $n \geq d$, $(I^n)^* = I^n$.

Proof. Let $k$ be as in Proposition 1.4(c) so that $(I^{n + m} : I^m) = I^m$ for all $n \geq k$, $m \in \mathbb{N}_g$ and let $d$ be such that $d(i) = \max \{1, k(i)\}$ for $1 \leq i \leq g$. The corollary now follows from Proposition 1.4(c) and Lemma 1.2(b). \hfill \Box

1.6. Proposition. (a) The set $\bigcup \{\text{Ass } R/I^n | n \in \mathbb{N}_g\}$ is finite.

(b) $\bigcup \{\text{Ass } R/(I^m) \cap \mathbb{N}_g | m \in \mathbb{N}_g\} \subseteq \bigcup \{\text{Ass } (I^n)^* | n \in \mathbb{N}_g\}$.

(c) If $\Delta \subseteq \{I^m | m \in \mathbb{N}_g\}$, then $\bigcup \{\text{Ass } R/(I^m) \cap \mathbb{N}_g | m \in \mathbb{N}_g\} \subseteq \bigcup \{\text{Ass } R/(I^n) | n \in \mathbb{N}_g\}$.

(d) If $I_1, \ldots, I_g$ are regular ideals, then $\bigcup \{\text{Ass } R/(I^m)^* | m \in \mathbb{N}_g\} \subseteq \bigcup \{\text{Ass } R/(I^n) \cap \mathbb{N}_g\}$.

Proof. Let $\mathcal{P} = R[I_1 t_1, \ldots, I_g t_g, t_1^{-1}, \ldots, t_g^{-1}]$ be the extended Rees ring of $R$ with respect to $I_1, \ldots, I_g$ and set $u_i = t_i^{-1}$. For $n \in \mathbb{N}_g$, $u^n \mathcal{P} \cap R = R^n$. Thus any $P \in \text{Ass } R/I^n$ lifts to a prime divisor $\mathcal{P}$ of $\mathcal{P}/u^n \mathcal{P}$. For some $1 \leq i \leq g$, $u_i \in \mathcal{P}$ and because each $u_i$ is regular, $\mathcal{P}$ must be a prime divisor of $u_i \mathcal{P}$. Now $\bigcup \{\text{Ass } \mathcal{P}/u^n \mathcal{P} | n \in \mathbb{N}_g\}$ is finite, so $\bigcup \{\text{Ass } R/I^n | n \in \mathbb{N}_g\}$ is finite as well. Thus we have (a).

For (b), by [2, Propositions 3.9 and 3.17], $\text{Ass } R/(I^m) \cap \mathbb{N}_g \subseteq A^*(I^m) = \text{Ass } R/I^{hm}$ for all large integers $h$. Hence part (b).
For (c), let \( m \in \mathbb{N}_g \) and \( P \in \text{Ass} R/(I^m)_{\Delta} \). We may write \( P = ((I^m)^{k_1} : x) = ((I^{m-k} : x)^{k_1} : x) = (I^{m-k} : x)^{k_1} : x) \), by the definition of \( \Delta \). Hence \( P \in \text{Ass} R/I^{m+k} \) and (c) follows.

For (d), we must show \( \text{Ass} R/(I^m)^* \subseteq \bigcup \{ \text{Ass} R/I^n \mid n \in \mathbb{N}_g \} \). Clearly it does no harm to assume \( m \geq (1, \ldots, 1) \) (since zero components can simply be ignored). Let \( \Delta = \{ n^* \mid n \in \mathbb{N}_g \} \). By Lemma 1.2(b), \( \text{Ass} R/(I^m)^* = \text{Ass} R/(I^m)_{\Delta} \subseteq \bigcup \{ \text{Ass} R/I^n \mid n \in \mathbb{N}_g \} \) by (c). \( \square \)

1.7. Theorem. Let \( \Delta \) be a multiplicatively closed set of non-zero ideals with \( \{ I^n \mid n \in \mathbb{N}_g \} \subseteq \Delta \). Then

(a) For any \( n, k \in \mathbb{N}_g \) satisfying \( n \geq k \), \( \text{Ass} R/(I^k)_{\Delta} \subseteq \text{Ass} R/(I^n)_{\Delta} \).

(b) If \( \bigcup \{ \text{Ass} R/(I^m)_{\Delta} \mid m \in \mathbb{N}_g \} \subseteq \bigcup \{ \text{Ass} R/(I^n) \mid n \in \mathbb{N}_g \} \), then for any sequence \( n_1 \leq n_2 \leq \cdots \) of elements from \( \mathbb{N}_g \), the sequence \( \text{Ass} R/(I^{n_1})_{\Delta} \subseteq \text{Ass} R/(I^{n_2})_{\Delta} \subseteq \cdots \) eventually stabilizes. In particular, there exists \( k \in \mathbb{N}_g \) such that \( \text{Ass} R(I^n)_{\Delta} \) is independent of \( n \) for all \( n \geq k \).

Proof. For (a), let \( P = ((I^k)_{\Delta} : x) \) belong to \( \text{Ass} R/(I^k)_{\Delta} \), with \( x \in R \). Writing \( (I^k)_{\Delta} = (I^kK : K) \) for some \( K \in \Delta \), we see that \( I^{n-k}(I^k)_{\Delta} = I^{n-k}(I^kK : K) \subseteq (I^nK : K) \subseteq (I^n)_{\Delta} \). Thus \( P = ((I^n)_{\Delta} : x) \subseteq (I^{n-k}(I^k)_{\Delta} : xI^{n-k}) \subseteq ((I^k)_{\Delta} : xI^{n-k}) \). However, we also claim that this last ideal is contained in \( P \). Let \( y \) belong to this ideal. Then \( yx \in ((I^k)_{\Delta} : I^{n-k}) \). For some \( L \in \Delta \), \( (I^k)_{\Delta} = (I^kL : L) \), so \( yx \in ((I^kL : L) : I^{n-k}) = (I^kI^{n-k}L : I^{n-k}L) \subseteq (I^n)_{\Delta} \) since \( I^{n-k}L \in \Delta \). Therefore \( y \in ((I^k)_{\Delta} : x) = P \) as desired. Thus \( P = ((I^n)_{\Delta} : xI^{n-k}) \), so \( P \in \text{Ass} R/(I^n)_{\Delta} \).

For (b), by Proposition 1.6 we have that \( \bigcup \{ \text{Ass} R/(I^n)_{\Delta} \mid n \in \mathbb{N}_g \} \) is finite, so if \( n_1 \leq n_2 \leq \cdots \), then by (a), \( \text{Ass} R/(I^n)_{\Delta} \subseteq \text{Ass} R/(I^{n_2})_{\Delta} \subseteq \cdots \) and this sequence must eventually stabilize. Now suppose that \( k = (k, \ldots, k) \in \mathbb{N}_g \) is such that \( \text{Ass} R/(I^k)_{\Delta} = \text{Ass} R/(I^{hk})_{\Delta} \) for all \( h \in \mathbb{N} \). (This follows from the \( g = 1 \) case of what was just shown.) For \( n \geq k \), select \( h \in \mathbb{N} \) such that \( hk \geq n \geq k \). Then by part (a), \( \text{Ass} R/(I^k)_{\Delta} \subseteq \text{Ass} R/(I^{hk})_{\Delta} = \text{Ass} R/(I^k)_{\Delta} \). \( \square \)

1.8. Corollary. Let \( I_1, \ldots, I_g \) be regular ideals.

(a) If \( n_1 \leq n_2 \leq \cdots \) is an increasing sequence from \( \mathbb{N}_g \), then the sequence \( \text{Ass} R/I_n \subseteq \text{Ass} R/I_{n_k} \subseteq \cdots \) eventually stabilizes. In particular, there exists \( k \in \mathbb{N}_g \) such that \( \text{Ass} R/I_n \) is independent of \( n \) for all \( n \geq k \).

(b) A similar statement holds for \( \text{Ass} R/(I^n)^* \), provided \( n_1 \geq (1, \ldots, 1) \).

(c) Let \( n_1 \leq n_2 \leq \cdots \) be an increasing sequence from \( \mathbb{N}_g \). Then the sequence \( \text{Ass} R/I_{n_1}, \text{Ass} R/I_{n_2}, \cdots \) eventually stabilizes. In particular, there exists \( k \in \mathbb{N}_g \) such that \( \text{Ass} R/I_n \) is independent of \( n \) for \( n \geq k \).

Proof. (a) follows from 1.1(b), 1.6 and 1.7 while (b) follows from 1.2(b), 1.6 and 1.7. For (c), we may suppose that for \( 1 \leq i \leq k \), \( \{ n_j(i) \mid j \geq 1 \} \) is infinite and for
k + 1 \leq i \leq g$, \( \{n_j(i) \mid j \geq 1 \} \) is finite. By ignoring small values of \( j \) we may assume that \((n_j(k + 1), \ldots, n_j(g)) = (s_1, \ldots, s_{g-k}) = s \in \mathbb{N}_{g-k}\). Let \( t_j \in \mathbb{N}_g \) be such that \( n_j = (t_j(1), \ldots, t_j(k), s_1, \ldots, s_{g-k}) \) and write \( I_{n_j} = A^j B^j \), where \( A^j = I_{t_j(1)}^j \cdots I_{t_j(k)}^j \) and \( B^j = I_{g-k}^{j-1} \cdots I_{g-k}^j \). Let \( A = \{A(t) \mid t \in \mathbb{N}_g\} \). Arguing as in the proof of \ref{1.7}(a), it is readily seen that \( \text{Ass } R/(I_{n_j})^j \subseteq \text{Ass } R/(I_{n_j})^j \subseteq \cdots \). On the other hand, \( (I_{n_j})^j = (A^j B^j)^j \) has the form \( (A^j B^j A^j') = A^j B^j \) for \( j \) large (by Proposition \ref{1.4}). Thus \( (I_{n_j})^j = (I_{n_j})^j \) for \( j \) large and part (c) now follows from \ref{1.6} and \ref{1.7}. \hfill \Box

2. The locally analytically unramified case

In this section we show that if \( R \) is locally analytically unramified with finite integral closure, then \( \Lambda \text{Ass } R/(I_{n_j})^j \) enjoys asymptotic stability for very general \( \Delta \)-closures. We also show that there exists a single \( K \in \Delta \) satisfying \( (I_{n_j})^j = (I_{n_j}^K : K) \) for all \( n \in \mathbb{N}_g \). This is accomplished by proving the following variation of the Artin-Rees lemma:

2.1. Lemma. Let \( I_1, \ldots, I_g \) be ideals of \( R \). For indeterminates \( t_1, \ldots, t_g \) set \( \mathcal{R} = R[I_1, \ldots, I_g] \) and \( \mathcal{R}_\Delta = R[\{(I_{n_j})^j \mid n \in \mathbb{N}_g\}] \). (Note that \( (I_{n_j})^j \cdot (I_{n+1}^m)^j \subseteq (I_{n+1}^m)^j \), so \( \mathcal{R}_\Delta \) is a ring and also an \( \mathcal{R} \)-module.) Then

(a) If \( \mathcal{R}_\Delta \) is a finite \( \mathcal{R} \)-module, there exists \( K \in \Delta \) such that for all \( n \in \mathbb{N}_g \), \( (I_{n_j})^j = (I_{n+1}^m)^j \). Also, there is an integer \( b \) such that if \( n \) and \( m \) are such that for all \( 1 \leq i \leq g \) either \( n(i) = m(i) \) or \( n(i) \geq m(i) \geq b \), then \( (I_{n_j})^j = (I_{n+1}^m)^j \). In particular, if \( n \geq m \geq b \), then \( (I_{n_j})^j = (I_{n+1}^m)^j \).

(b) If there is a regular ideal \( K \in \Delta \) such that \( (I_{n_j})^j = (I_{n+1}^m)^j \) for all \( n \in \mathbb{N}_g \), then \( \mathcal{R}_\Delta \) is a finite \( \mathcal{R} \)-module.

Proof. For (a), the hypothesis implies that there exist finitely many \( m_j \in \mathbb{N}_g \) such that \( \mathcal{R}_\Delta = \sum \mathcal{R}((I_{n_j})^j t_{m_j}) \) over \( 1 \leq j \leq r \).

For each \( j \), there is a \( K_j \in \Delta \) such that \( (I_{n_j})^j = (I_{n_j}^m)^j \). Let \( K \) be the product of the \( K_j \) over all \( 1 \leq j \leq r \). Then \( (I_{n_j})^j = (I_{n_j}^m)^j \) for all \( j \).

Now, consider the submodule \( \mathcal{I} \) of \( \mathcal{R}_\Delta \) having the form \( \sum (I_{n_j}^m : K)^n \) over all \( n \in \mathbb{N}_g \). (Since for \( m \in \mathbb{N}_g \), \( I_{n_j}^m (I_{n_j}^m : K) \subseteq (I_{n+1}^m)^j : K \), this is a submodule.) Since for \( 1 \leq j \leq r \), \( \mathcal{I} \) contains \( (I_{n_j}^m : K)^n = (I_{n_j}^m)^j t_n \), and these last sets generate \( \mathcal{R}_\Delta \) over \( \mathcal{R} \) as an \( \mathcal{R} \)-module, we see that \( \mathcal{I} = \mathcal{R} \). It follows that \( (I_{n_j}^m : K) = (I_{n_j}^m) \) for all \( n \in \mathbb{N}_g \). This proves the first part of (a).

Now let \( b = \max\{m_j(i) \mid 1 \leq j \leq r, 1 \leq i \leq g \} \). Suppose that \( n \) and \( m \) are such that for each \( 1 \leq i \leq g \), either \( n(i) = m(i) \) or \( n(i) \geq m(i) \geq b \). Since \( \mathcal{R}_\Delta = \sum \mathcal{R}((I_{n_j})^j t_{m_j}) \) over \( 1 \leq j \leq r \), looking at the \( t_n \)th term in \( \mathcal{R}_\Delta \), we see that \( (I_{n_j})^j = \sum (I_{n+1}^m)^j (I_{n+1}^m) \) over those \( 1 \leq j \leq r \) with \( m_j \leq n \). A similar statement can be made about \( (I_{n_j})^j \).

However, we claim that \( m_j \leq n \) if and only if \( m_j \leq m \). This follows from the fact that in the \( i \)th component, either \( m(i) = n(i) \) or both \( m(i) \) and \( n(i) \) are at least as large as \( b \), which in turn is at least as large as \( m_j(i) \). Therefore, the summations for
(I^m)_\Delta and (I^n)_\Delta involve exactly the same set of j, and, in fact differ only in that the first has I^{m-m_j} appearing in the place where the second has I^{n-m_j} appearing. Clearly n \geq m so I^{n-m_j} = I^{m-m_j}. The second part of (a) follows from this.

For (b), suppose that K is a regular ideal in \Delta and that (I^n)_\Delta = (I^nK : K) for all n \in \mathbb{N}_g. Then K(I^n)_\Delta \subseteq I^nK \subseteq I^n, and so, K\mathcal{R}_\Delta \subseteq \mathcal{R}. Since K contains a regular element x of R (which remains regular in \mathcal{R}), we see that \mathcal{R}_\Delta \subseteq \mathcal{R}x^{-1}. Thus \mathcal{R} is a finite \mathcal{R}-module, since \mathcal{R} is Noetherian. □

2.2. Theorem. Let I_1, \ldots, I_g be regular ideals. Assume that R is a locally analytically unramified ring with finite integral closure. Let \Delta be any multiplicatively closed set of regular ideals such that \{I^m \mid m \in \mathbb{N}_g\} \subseteq \Delta. Then for \mathcal{R} and \mathcal{R}_\Delta as in Lemma 2.1:

(a) \mathcal{R}_\Delta is a finite \mathcal{R}-module.
(b) There exists K \in \Delta, such that (I^n)_\Delta = (I^nK : K) for all n \in \mathbb{N}_g.
(c) \bigcup \{\text{Ass } R/(I^n)_\Delta \mid n \in \mathbb{N}_g\} is a finite set.
(d) If n_1 \leq n_2 \leq \cdots is an increasing sequence of elements from \mathbb{N}_g, then the sequence of sets \text{Ass } R/(I^n)_\Delta \subseteq \text{Ass } R/(I^n)_\Delta \subseteq \cdots eventually stabilizes. In particular, there exists a k \in \mathbb{N}_g such that \text{Ass } R/(I^n)_\Delta is independent of n for all n \geq k.

Proof. By [1, Lemma 1], \mathcal{R}_\Delta is a finite \mathcal{R}-module. Thus (a) holds and (b) follows from the proof of Theorem 1.7, once we prove (c). For this let \mathcal{P} = \mathcal{R}[t_1^{-1}, \ldots, t_g^{-1}] and \mathcal{P}_\Delta = \mathcal{R}_\Delta[t_1^{-1}, \ldots, t_g^{-1}]. Then \mathcal{P}_\Delta is a finite \mathcal{P}-module, and is therefore a Noetherian ring. Since \mathcal{P}(n) \cap K = (I^n)_\Delta for all n \in \mathbb{N}_g, any P \in \text{Ass } R/(I^n)_\Delta lifts to an element of \text{Ass } \mathcal{P}_\Delta/I^n\mathcal{P}_\Delta. Since \bigcup \{\text{Ass } \mathcal{P}_\Delta/I^n\mathcal{P}_\Delta \mid n \in \mathbb{N}_g\} is finite (as in the proof of Proposition 1.6), \bigcup \{\text{Ass } R/(I^n)_\Delta \mid n \in \mathbb{N}_g\} is finite, and the proof is complete. □

2.3. Corollary. Let R be as above and I_1, \ldots, I_g regular ideals. Then there is an integer k such that for all n \in \mathbb{N}_g, (I^n)^k = ((I^n)^{k+1} : (I^n)^k).

Proof. We will find an integer k(g) which satisfies the conclusion of the result for all n \geq (1, \ldots, 1). If n has some zero components, then we will delete those I_j for which n(i) = 0, and so will simply have a smaller value of g to deal with. Thus, the final k we take will be the maximum of the k(d) over 1 \leq d \leq g.

Assume n \geq (1, \ldots, 1). Then (I^n)^* = (I^n)_\Delta by Lemma 1.2(b), assuming \Delta = \{I^m \mid m \in \mathbb{N}_g\}. By Theorem 2.2(b), there is an I^* \in \Delta such that (I^n)_\Delta = (I^n + c : I^*) for all n \in \mathbb{N}_g. Let k(g) equal the maximum component of c. By Lemma 1.2, (I^n + c : I^*) \subseteq ((I^*)^{k(g)+1} : (I^n)^{k(g)}). Thus (I^n)^* \subseteq ((I^n)^{k(g)+1} : (I^n)^{k(g)}), and the reverse inclusion is by the definition of (I^n)^*. □

We close by mentioning two questions we have been unable to answer.
Question 1. If $R$ is an arbitrary Noetherian ring and $\Delta$ a multiplicatively closed set of regular ideals containing $\{I^m \mid m \in \mathbb{N}_g\}$, do the sets $\text{Ass } R/(R^m)_{\Delta}$ enjoy asymptotic stability? If this always holds for $g = 1$ and $I_1 = (b)$, $b$ a regular element, then the answer is yes. In fact it is enough to know that $\bigcup \{\text{Ass } R/(b^n)_{\Delta} \mid n \geq 1\}$ is finite.

Question 2. For which multiplicatively closed sets of ideals $\Delta$ does it hold that $\bigcup \{\text{Ass } R/(R^m)_{\Delta} \mid m \in \mathbb{N}_g\} \subseteq \bigcup \{\text{Ass } R/I^n \mid n \in \mathbb{N}_g\}$?

References